On Impact of Earth Constraint on TDOA-Based Localization Performance in Passive Multisatellite Localization Systems

Feng Shu, Shuping Yang, Jinhui Lu, and Jun Li

Abstract—**In a passive multisatellite location system with Earth constraint (EC), the localization problem of time differences of arrival (TDOAs) is formulated as a quadratic optimization (QO) problem with two equality constraints: EC and variable constraint (VC). Which constraint is more important? To evaluate the importance of the two constraints, the original QO problem with both EC and VC is relaxed into two individual QO problems with only one equality constraint EC or VC being kept. A convex combination operation is performed on the two optimal solutions associated with the two relaxed QO problems to form a new weighted solution. Numerical simulation results show that the new weighted coefficient can achieve the Cramer–Rao lower bound with EC as the weighted factor of the solution exploiting only VC tends to zero, and increasing the value of this weighted coefficient will gradually degrade the overall localization performance. Thus, we conclude that the EC plays an extremely important role in improving the performance of TDOA-based localization.**

Index Terms—**Convex combination, Earth constraint (EC), passive source localization, quadratic optimization (QO), time differences of arrival (TDOAs), variable constraint (VC).**

I. Introduction

A passive source localization plays an extremely important role in many applications such as unmanned aerial vehicle networks, wireless sensor networks, navigation, microphone arrays, mobile communications, and tracking of acoustic sources [1]. In the past thirty years, research efforts mainly concentrated on developing various methods for a source position estimation. To estimate the positions of stationary emitters and sensors, it usually needs to measure time differences of arrival (TDOAs), times of arrival (TOAs), or angles of arrival (AOAs) of the passive source to spatially separated sensors [2]–[6]. To achieve a high-precision localization, TDOA only needs perfect time synchronization between satellite stations; however, TOA requires a perfect time synchronization between passive target and satellite stations, which is hard to be realized due to unknown passive target information. Thus, compared to TOA, the TDOA-based location can reduce system complexity and be more easily implemented. Usually, the precision of TDOA measurement is higher than that of AOA, the location accuracy of TDOA-based location could be higher. Hence, the kind of methods based on TDOA is more popular than other kinds. The main corresponding methods used are least squares in[2], Taylor-series estimation in [3], Hough transform in[4], particle swarm optimization in[5], semidefinite relaxation techniques in [6], and so on.

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In this paper, we still focus on the investigation of the TDOA-based localization in passive multisatellite systems. In [7], a near-closed-form solution is proposed where only Earth constraint (EC) is exploited, while the variable constraint (VC) is simply and directly abandoned. Unlike [7], in this paper, we are mainly concerned with which constraint of EC and VC is more important and the feasibility of ignoring VC instead of a detailed localization algorithm. In Section III, by taking both EC and VC into account, the problem of TDOA-based localization in a passive multisatellite location system is cast as a quadratic optimization (QO) problem with two equality constraints: EC and VC. Then, the optimization problem with two constraints is relaxed into two independent optimization problems with each keeping only one constraint. The associated optimal solutions to the two optimization subproblems are synthesized to a new solution by a convex combination. The newly formed solution is used to check which constraint is more important by varying weighted coefficients. From simulation results and analysis, we claim that the EC plays a dominant role in localization performance compared to VC. By using the fact, with only EC, the proposed approximate analytic solution in [7] may be adopted to provide a low-complexity localization. Conversely, with two constraints VC and EC, there exists no closed-form solution. Then, the semidefinite programming method in [8] is an efficient and feasible way to solve the original optimization problem of TDOA localization, and its computational complexity of TDOA localization has the order $O(n^{4.5}\log(1/\epsilon))$ floating point operations (FLOPs), which is far higher than that $O(n^3)$ FLOPs of the method [7]. Hence, the system complexity is significantly reduced.

Notations: Bold letters denote vectors and matrices. $(\cdot)^T$ denotes the transposition. $\|\cdot\|$ denotes the norm-2 of a vector. The notation $\mathbf{v}(1:n)$ denotes a column vector consisting of the first n elements of column vector $\mathbf{v}$.

II. System Model

In a passive multisatellite localization, the position of passive target $\mathcal{S}$ is denoted by $\mathbf{r} = (x, y, z)^T$, and there are M satellites $\mathcal{T}_1, \ldots, \mathcal{T}_M$ with positions $\mathbf{r}_1 = (x_1, y_1, z_1)^T, \ldots, \mathbf{r}_M = (x_M, y_M, z_M)^T$, respectively. Without loss of generality, the first satellite $\mathcal{T}_1$ is viewed as the primary one and the remaining $M-1$ satellites are the secondary ones. In the presence of receiver noise, the measured TDOA between $\mathcal{T}_1$ and $\mathcal{T}_i$, $i = 2, \ldots, M$ is modeled as

$$\hat{\tau}_{i1} = \tau_{i1} + \Delta\tau_{i1} = \frac{d_{si} - d_{s1}}{c} + \Delta\tau_{i1} \tag{1}$$

where c is the speed of light and $\Delta\tau_{i1}$ is the TDOA measurement error, which is approximately modeled as a Gaussian random variable with zero mean and variance σ^2. d_{si} denotes the distance between $\mathcal{S}$ and $\mathcal{T}_i$, i.e., $d_{si} = \| \mathbf{r} - \mathbf{r}_i \|$.

Rearranging and squaring both sides of (1) and using the expression of d_{si}, we obtain

$$\begin{aligned}(c\hat{\tau}_{i1})^2 &+ \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_i^T\mathbf{r}_i \\ &= 2\left(\mathbf{r}_1^T - \mathbf{r}_i^T\right)\mathbf{r} - 2\left(c\hat{\tau}_{i1}d_{s1}\right) + n_{i-1}\end{aligned} \tag{2}$$

where $i \in \{2, 3, \ldots, M\}$, and $n_{i-1} \triangleq 2c\Delta\tau_{i1}d_{si} + c^2\Delta\tau_{i1}^2$. Collecting all the equations like (2) yields the following matrix-form system of localization equations as

$$\mathbf{b} = \mathbf{A}\mathbf{w} + \mathbf{n} \tag{3}$$

where the vector $\mathbf{b}$ is defined as

$$\left(c^2\hat{\tau}_{21}^2 + \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_2^T\mathbf{r}_2, \ldots, \quad c^2\hat{\tau}_{M1}^2 + \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_M^T\mathbf{r}_M \right)^T$$

$$\mathbf{A} \triangleq \begin{pmatrix} 2\mathbf{r}_1^T - 2\mathbf{r}_2^T & -2c\hat{\tau}_{21} \\ 2\mathbf{r}_1^T - 2\mathbf{r}_3^T & -2c\hat{\tau}_{31} \\ \vdots & \\ 2\mathbf{r}_1^T - 2\mathbf{r}_M^T & -2c\hat{\tau}_{M1} \end{pmatrix}$$

$$\mathbf{w} \triangleq \left(\mathbf{r}^T, d_{s1}\right)^T$$

$$\mathbf{n} \triangleq (n_1, n_2, \ldots, n_{M-1})^T$$

where $d_{s1}^2 = (\mathbf{r}_1 - \mathbf{r})^T(\mathbf{r}_1 - \mathbf{r})$, which is called VC in the following. Considering that passive source $\mathcal{S}$ is on the Earth surface, the EC can be given by $\mathbf{r}^T\mathbf{r} = r_e^2$, where r_e is the Earth radius [9]. In what follows, we will apply the constrained Cramer–Rao lower bound (CRLB) of TDOA localization in [9] to evaluate our proposed method.

III. Problem Formulation and Proposed Weighted Method

In this section, we model the TDOA location problem as a QO with two quadratic constraints by a variable substitution method. Then, we relax the original optimization problem into two solvable optimization problems with approximate analytic expression. Finally, we combine the optimal solutions to the two relaxed optimization problems into a new weighted solution. In the next simulation section, the new solution will be adopted to evaluate the importance of the two constraints.

A. Problem Formulation

For the convenience of derivation below, let us define

$$\widetilde{\mathbf{w}} \triangleq \left(\mathbf{r}^T - \mathbf{r}_1^T, d_{s1}\right)^T \quad \widetilde{\mathbf{r}}_1 \triangleq \left(\mathbf{r}_1^T, 0\right)^T \quad \widetilde{\mathbf{b}} \triangleq \mathbf{b} - \mathbf{A}\widetilde{\mathbf{r}}_1. \tag{4}$$

By means of the above-mentioned variable substitution, (3) is transformed into

$$\widetilde{\mathbf{b}} = \mathbf{A}\widetilde{\mathbf{w}} + \mathbf{n}. \tag{5}$$

Two constraints VC and EC in Section II are rewritten in matrix form

$$\left(\mathbf{r}^T - \mathbf{r}_1^T, d_{s1}\right)\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & d_{s1} \end{pmatrix}\begin{pmatrix} \mathbf{r} - \mathbf{r}_1 \\ d_{s1} \end{pmatrix} = 0$$

and

$$\begin{aligned}&\left(\mathbf{r}^T - \mathbf{r}_1^T, d_{s1}\right)\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{r} - \mathbf{r}_1 \\ d_{s1} \end{pmatrix} \\ &+ 2\left(\mathbf{r}^T - \mathbf{r}_1^T, d_{s1}\right)\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{r} - \mathbf{r}_1 \\ 0 \end{pmatrix} = r_e^2 - \mathbf{r}_1^T\mathbf{r}_1\end{aligned}$$

respectively. Using the new variables in (4), the VC and EC are simplified as

$$\widetilde{\mathbf{w}}^T\mathbf{\Sigma}_s\widetilde{\mathbf{w}} = 0 \tag{6}$$

and

$$\widetilde{\mathbf{w}}^T\mathbf{\Sigma}_e\widetilde{\mathbf{w}} + 2\widetilde{\mathbf{w}}^T\mathbf{\Sigma}_e\widetilde{\mathbf{r}}_1 = \rho \tag{7}$$

respectively, where $\rho = r_e^2 - x_1^2 - y_1^2 - z_1^2$, $\mathbf{\Sigma}_s = \text{diag}\{1, 1, 1, -1\}$, and $\mathbf{\Sigma}_e = \text{diag}\{1, 1, 1, 0\}$. By using (5), VC in (6), and EC in (7), we establish the following QO problem:

$$\begin{aligned}\min_{(\widetilde{\mathbf{w}})} \quad & \left(\widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{w}}\right)^T\left(\widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{w}}\right) \\ \text{s.t.} \quad & \widetilde{\mathbf{w}}^T\mathbf{\Sigma}_s\widetilde{\mathbf{w}} = 0\,(\text{VC}) \\ & \widetilde{\mathbf{w}}^T\mathbf{\Sigma}_e\widetilde{\mathbf{w}} + 2\widetilde{\mathbf{w}}^T\mathbf{\Sigma}_e\widetilde{\mathbf{r}}_1 = \rho\,(\text{EC}).\end{aligned} \tag{8}$$

Clearly, in the optimization problem mentioned above, there are a quadratic objective function and two quadratic equality constraints. Since both matrix $\mathbf{\Sigma}_s$ in the first constraint of the above-mentioned optimization and its negative matrix $-\mathbf{\Sigma}_s$, with both negative and positive eigenvalues (-1 and $+1$), are not positive semidefinite matrices, the QO (8) is nonconvex. How to find an efficient way to solve the nonconvex QO problem mentioned above is still an open problem [10].

B. Proposed Weighted Method

It is hard to solve the nonconvex QO problem in (8) due to its two equality constraints[11]. In this section, two relaxed optimization problems by deleting any equality constraint in (8) are formed and their two approximate closed-form solution can be solved by the method proposed in [7]. Finally, we perform a combining operation on the two solutions to yield a weighted solution.

Removing the EC in (8) yields the following first relaxed optimization problem:

$$\begin{aligned}\min_{(\widetilde{\mathbf{w}})} \quad & \left(\widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{w}}\right)^T\left(\widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{w}}\right) \\ \text{s.t.} \quad & \widetilde{\mathbf{w}}^T\mathbf{\Sigma}_s\widetilde{\mathbf{w}} = 0\end{aligned} \tag{9}$$

which is not a convex optimization but solvable as indicated in [7] and [11]. Removing the VC in (8) gives the following second relaxed optimization problem:

$$\begin{aligned}\min_{(\widetilde{\mathbf{w}})} \quad & \left(\widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{w}}\right)^T\left(\widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{w}}\right) \\ \text{s.t.} \quad & \widetilde{\mathbf{w}}^T\mathbf{\Sigma}_e\widetilde{\mathbf{w}} + 2\widetilde{\mathbf{w}}^T\mathbf{\Sigma}_e\widetilde{\mathbf{r}}_1 = \rho\end{aligned} \tag{10}$$

which is a convex optimization due to $\mathbf{\Sigma}_e$ being a positive semidefinite matrix. Utilizing the Lagrange multiplier method mentioned in [7] and [11], the optimal solutions for (9) and (10) are denoted as $\widetilde{\mathbf{w}}_{\text{VC}}$ and

TABLE I
SATELLITE POSITIONS

Satellite	Height h	Longitude α	Latitude β
1	1200	130°	0.0°
2	1126.1	130.70°	0.7013°
3	1126.1	130.70°	−0.7013°
4	1126.1	129.30°	0.7013°
5	1126.1	129.30°	−0.7013°

$\widetilde{\mathbf{w}}_{\text{EC}}$, respectively. Using (4), the estimated target positions are

$$\mathbf{r}_{\text{VC}} = \widetilde{\mathbf{w}}_{\text{VC}}(1:3) + \mathbf{r}_1 \text{ and } \mathbf{r}_{\text{EC}} = \widetilde{\mathbf{w}}_{\text{EC}}(1:3) + \mathbf{r}_1. \quad (11)$$

Performing a convex combination of the above-mentioned two solutions yields the estimated target position of the form

$$\widehat{\mathbf{r}} = \lambda_1 \mathbf{r}_{\text{VC}} + \lambda_2 \mathbf{r}_{\text{EC}} \quad \lambda_1 + \lambda_2 = 1 \quad \lambda_1,\, \lambda_2 \geq 0. \quad (12)$$

Given the two weighted coefficients, the corresponding target position is readily obtained. However, what are the optimal values for λ_1 and λ_2? In the next section, we gain some valuable insight into this problem from numerical simulations.

IV. SIMULATIONS AND DISCUSSIONS

In this section, simulations conducted by MATLAB are used to evaluate the influence of the change in weighted coefficients on the performance of TDOA-based localization in three-dimensional space and the CRLB with EC is used as reference. Similar to [7] and [12], we choose five satellites in our simulation. Satellite 1 is used as the primary satellite and the remaining satellites as the secondary ones. For convenience of simulation and computation mentioned below, five satellites form a rectangular symmetric pyramid constellation and their positions are listed in Table I, where h is the height from satellite/target to ground in terms of kilometers (km), α denotes the longitude, and β denotes the latitude. Then, the target position $\mathbf{r} = (x, y, z)^T$ is readily expressed by its spherical coordinates $(\alpha,\, \beta,\, r_e + h)^T$ with the Earth radius r_e being 6400 km.

Given the target position is fixed at $\mathbf{r} = (0, 132°, 2°)$, Fig. 1 demonstrates the curves of RMSE versus λ_1 for $\sigma = 1$, 20, and 50 ns. From this figure, it is evident that the performance of the proposed weighted method approximately degrades with the increase in λ_1 from 0 to 1 for all three different TDOA measurement errors. When λ_1 is close to zero, the proposed method can achieve the optimal performance, i.e., the CRLB with EC. For example, at $\lambda_1 = 1$, only the VC works. The performance of the proposed method is far worse than the CRLB with EC. Otherwise, at $\lambda_1 = 0$, only the EC works. This means that the VC makes no contribution to the localization performance due to $\lambda_1 = 0$. Here, the performance of the proposed method is very close to the CRLB with EC. Thus, we conclude that, compared to VC, EC plays a dominant role in improving the localization performance. Observing the subfigure at the upper-left corner of Fig. 1, it is apparent that the proposed method achieves the CRLB with EC for $\lambda_1 \in [0,\, 0.015]$.

Given the TDOA measurement error variance $\sigma = 20$ ns, Fig. 2 illustrates the curves of RMSE versus λ_1 for three different target positions $\mathbf{r} = (0, 134°, -4°)$, $(0, 140°, 10°)$, and $(0, 110°, 12°)$ on the Earth surface, respectively, which are 500, 1500, and 2500 km far from the subastral point, which is defined as the intersection point between the Earth surface and the line of the primary satellite to the Earth center. We find the same fact that the VC is still dominated by the EC regardless

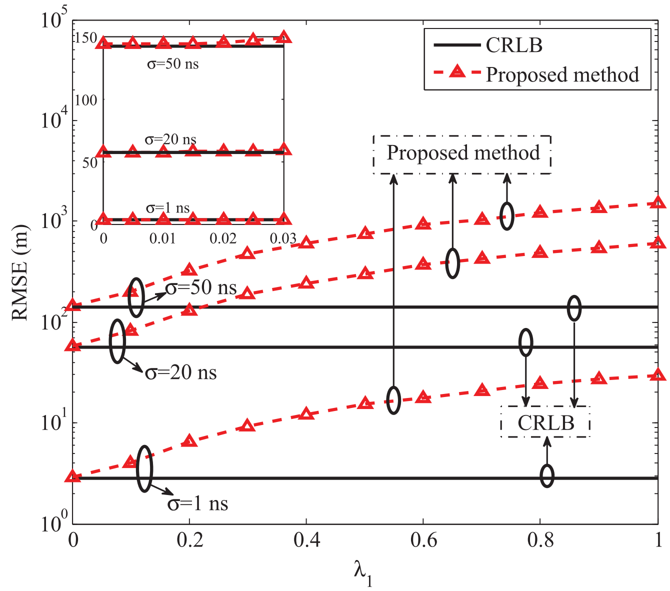

Fig. 1. Curves of RMSE versus λ_1 for three different TDOA measurement error variances.

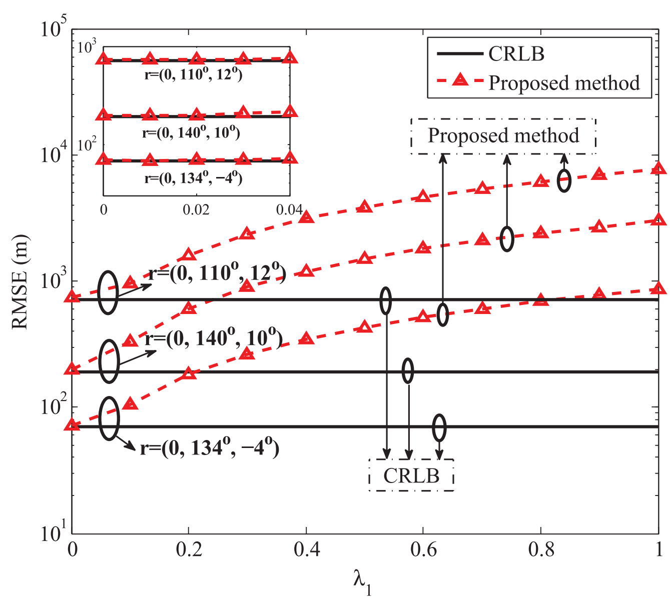

Fig. 2. Curves of RMSE versus λ_1 for three different target positions.

of the values of target position. This implies VC can be removed from the constraints of the optimization problem in (8).

To further verify the weight of VC and EC, all satellite positions are moved to the new positions listed in Table II. Figs. 3 and 4 illustrate the curves of RMSE versus λ_1 for three distinct variances of measurement errors and three different target positions, respectively. From Figs. 3 and 4, it follows that for the infinitely small values of λ_1, for example, $\lambda_1 < 0.02$, meaning only EC exists, the proposed method can reach the CRLB with EC. In summary, the dominant position of EC is very evident compared to VC.

Now, we will explain why EC is important from another aspect. In terms of the relationship between spherical and rectangular coordinates,

TABLE II
SATELLITE POSITIONS

Satellite	Height h	Longitude α	Latitude β
1	1000	130°	0.0°
2	926.16	130.72°	0.7205°
3	926.16	130.72°	−0.7205°
4	926.16	129.27°	0.7205°
5	926.16	129.27°	−0.7205°

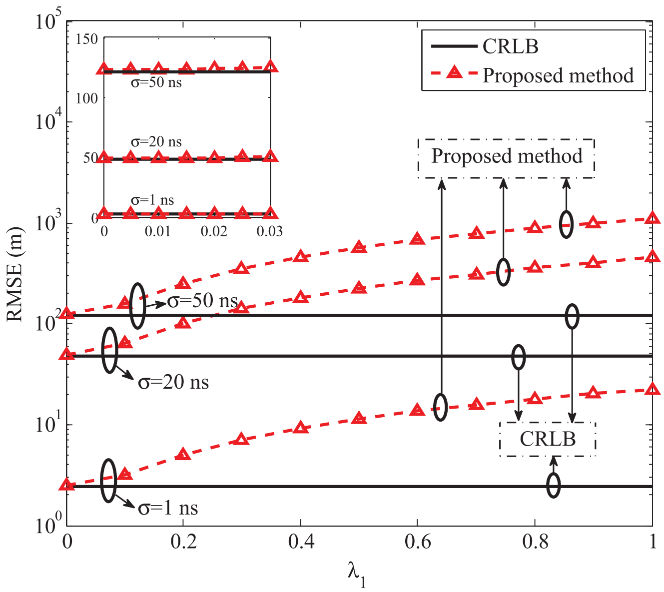

Fig. 3. Curves of RMSE versus λ_1 for three different TDOA measurement error variances with new satellite positions.

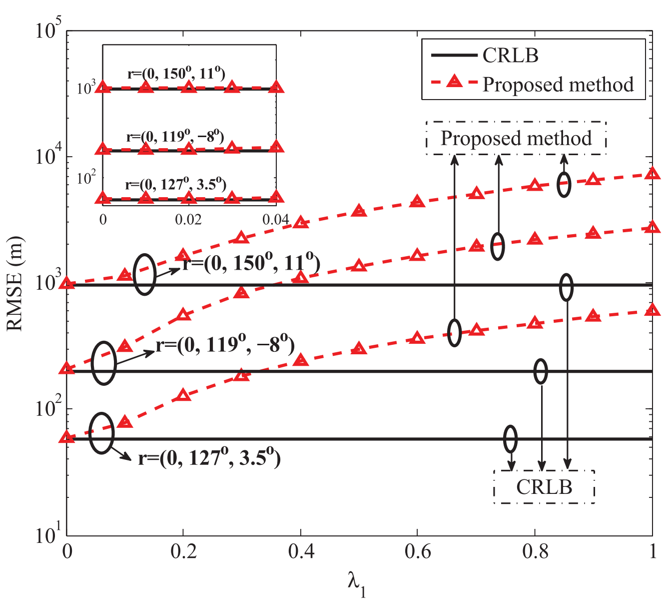

Fig. 4. Curves of RMSE versus λ_1 for three different target positions with new satellite positions.

it is apparent that all three coordinates x, y, and z of target position are linear functions of altitude h. EC means that target is confined to the Earth surface, i.e., $h = 0$. This implies that there is no measurement error along altitude h and thus further reduces the number of factors affecting the estimate performance of x, y, and z. Hence, the EC can significantly improve the localization performance.

V. CONCLUSION

In this paper, a weighted method is proposed for a passive multi-satellite location system based on TDOA measurements by exploiting both VC and EC. Then, we make an investigation of which one of two equality constraints VC and EC is more important via the Monte-Carlo method. Simulation shows that the performance of the proposed weighted method converges to the CRLB with EC as the weighted coefficient for the solution of the relaxed optimization problem with VC decreases from one to zero. Therefore, we conclude that, compared to VC, EC has a dominant impact on localization performance. In other words, the VC can be deleted from the original optimization problem without affecting the overall localization performance. This result will simplify the original optimization problem. In other words, it will substantially reduce the system design and computational complexity for the future passive multisatellite TDOA-based localization.

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