# Power Allocation in the High SNR Regime for a Multicast Cell with Regenerative Network Coding

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***Abstract*—This letter focuses on power allocation schemes for a basic multicast cell with wireless regenerative network coding (RNC). In RNC, mixed signals received from the two sources are jointly decoded by the relay where decoded symbols are superposed in either the complex field (RCNC) or Galois field (RGNC) before being retransmitted. We deduce the optimal statistical channels state information (CSI) based power allocation and give a comparison between the two RNCs. When instantaneous CSI is available at each transmitter, we propose a suboptimal power allocation for RCNC, which achieves better performance.**

***Index Terms*—Wireless network coding, multicast network, power allocation, frame error probability.**

## I. Introduction

Recently, how to leverage network coding [1] in wireless networks to improve system capacity has drawn increasing interest [2]-[6]. However, these works focus on the multi-access model or unicast model. Since multicast topology is popular in practical wireless networks, it is desirable to investigate network coding in a basic wireless multicast cell.

Fig. 1 depicts a basic multicast cell with 2 sources, 1 relay and 2 destinations ($2-1-2$ model). Suppose that both $s_1$ and $s_2$ transmit their messages to the same destination set $\{d_1, d_2\}$ simultaneously. However, $d_1$ (or $d_2$) is out of the transmission range of $s_2$ (or $s_1$). The shared relay can help $s_1$ (or $s_2$) to reach $d_2$ (or $d_1$). When wireless network coding is applied to the relay, the transmission process takes two time slots, i.e.,

1. $s_1 \rightarrow \{r,\ d_1\}$ with $X_{s_1}$; $s_2 \rightarrow \{r,\ d_2\}$ with $X_{s_2}$,
2. $r \rightarrow \{d_1,\ d_2\}$ with $f(X_{s_1}, X_{s_2})$,

where $f(\cdot)$ denotes the network coding protocol. In non-regenerative network coding, the mixed signals from the two sources are not decoded at the relay before retransmission to the destinations [7], while in regenerative network coding (RNC), joint maximum likelihood (ML) decoder is performed at the relay. Then the decoded symbols are superposed in either the complex field (RCNC) or Galois field (RGNC) before being retransmitted by the relay. In this letter, we propose the statistical and instantaneous CSI based power allocation schemes in the high signal-to-noise ratio (SNR) regime to improve the system performance in terms of system frame error probability (SFEP). Throughout this letter, we use the following notation: $\bar{k}$ denotes the complementary element of the number $k$ in the set $\{1,\ 2\}$. $\hat{x}$ denotes a decoder's estimate of the symbol $x$. $\mathcal{E}(\cdot)$ is the statistical expectation. $z(\rho) \triangleq \mathcal{O}(y(\rho))$, for $y(\rho) > 0$, means that there is a positive constants $c$ such that $|z(\rho)| \leq cy(\rho)$ when $\rho$ is large enough.

Manuscript received October 15, 2008. The associate editor coordinating the review of this letter and approving it for publication was A. Sprintson.

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This work is supported by NSF China #60672067, by PUJIANG Talents #07PJ4046, and by Huawei Fund #YJCB2008048WL.

Digital Object Identifier 10.1109/LCOMM.2009.081694

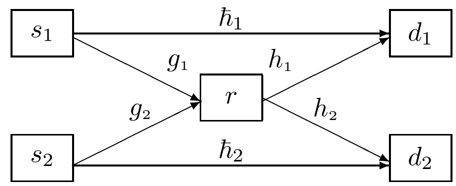


Fig. 1. $2-1-2$ wireless multicast system.

## II. System Model

Channel coefficients shown in Fig. 1 are assumed to have a Rayleigh distribution with zero mean and unit variance. The noises observed by all the receivers are assumed to have a Gaussian distribution with zero mean and variance $\sigma^2$. We denote $P$ as the average total network transmission power over a time slot. Then the system SNR is defined as $\rho \triangleq \frac{P}{\sigma^2}$.

We define $\mathbf{x}_s \triangleq [x_{s_1}, x_{s_2}]$ as a system frame where $x_{s_k}$ is transmitted by $s_k\,(k \in \{1,2\})$. The decoded frame to be transmitted by the relay $r$ in the second time slot is denoted as $\mathbf{x}_r = [x_{r_1}, x_{r_2}]$. All symbols in $\mathbf{x}_s$ and $\mathbf{x}_r$ are $i.i.d$ and selected from the same $2^R$-QAM constellation set $\mathcal{Q}$ with zero mean and variance $2P$. The signal received by $d_k$ in the first time slot is $y_{d_k,1} = \hbar_k\sqrt{\kappa_k}x_{s_k} + v_{d_k,1}$ where $\kappa_k$ is the power allocation factor (PAF) for $x_{s_k}$. However, in the second time slot, it is different between the two protocols, i.e.,

$$\begin{aligned} y_{d_k,2} &= h_k(\sqrt{\tau_1}x_{r_1} + \alpha\sqrt{\tau_2}x_{r_2}) + v_{d_k,2} && \text{for RCNC,} \\ y_{d_k,2} &= h_k\sqrt{\tau}x_r + v_{d_k,2} && \text{for RGNC,} \end{aligned} \tag{1}$$

where $\alpha = e^{\frac{3j\pi}{4}}$ is the precoder used to achieve full diversity gain [3], $x_r \in \mathcal{Q}$ is the superposition of $x_{r_1}$ and $x_{r_2}$ in Galois field, $\sqrt{\tau_k}$ and $\sqrt{\tau}$ are the PAFs of $x_{r_k}$ and $x_r$ respectively, and $v_{d_k,l}\,(l \in \{1,2\})$ is the noise observed by $d_k$ in the $l$-th time slot. To compare the two RNC protocols fairly, we let $\tau = \tau_1 + \tau_2$. Then we have $\mathcal{E}(\tau_1|x_{r_1}|^2 + \tau_2|x_{r_2}|^2) = \mathcal{E}(\tau|x_r|^2)$. Define $\kappa \triangleq \kappa_1 + \kappa_2$, and then $\kappa + \tau = 1$. So the total power consumed during a frame period, i.e., two time slots, is

$$\mathcal{E}_{\mathbf{x}_s,\mathbf{x}_r}\left(\kappa_1|x_{s_1}|^2 + \kappa_2|x_{s_2}|^2 + \tau|x_r|^2\right) = 2P. \tag{2}$$

Note that ML decoding is performed at all receivers. In RCNC protocol, if $r$ can successfully decode $\mathbf{x}_s$, i.e., $\mathbf{x}_r = \mathbf{x}_s$, then after the second time slot, the joint ML decoder at $d_k$ is

$$\begin{aligned} (\hat{x}_{s_1}, \hat{x}_{s_2})_{d_k} = \arg\min_{x_{s_1},x_{s_2}\in\mathcal{Q}} \big\{ &|y_{d_k,1} - \hbar_k\sqrt{\kappa_k}x_{s_k}|^2 \\ &+ |y_{d_k,2} - h_k(\sqrt{\tau_1}x_{s_1} + \alpha\sqrt{\tau_2}x_{s_2})|^2 \big\}. \end{aligned} \tag{3}$$

While in RGNC protocol, $d_k$ decodes $x_{s_k}$ after the first time slot and decodes $x_r$ after the second time slot since the two symbols are mutually independent. Then $x_{s_{\bar{k}}}$ can be worked out by Galois field operation between $x_{s_k}$ and $x_r$.

## III. Power Allocation Schemes

We suppose that when $\mathbf{x}_s$ is wrongly decoded by either of the destinations, a system frame error event (SFEE) occurs. Then SFEP is defined as the probability of SFEE, which can be expressed as $P_{sys} = P_{d_1}(1-P_{d_2})+P_{d_2}(1-P_{d_1})+P_{d_1}P_{d_2}$, where $P_{d_k}$ is the FEP of $d_k$, i.e., the probability of the event that $d_k$ wrongly decodes $\mathbf{x}_s$. According to the system model, if $r$ wrongly decodes $\mathbf{x}_s$, there is at least one destination $d_k$ which can not extract the right symbol $x_{s_{\bar{k}}}$ from $r$ and thus SFEE occurs with probability 1. Then $P_{sys}$ is rewritten as

$$P_{sys} = P_r + (1-P_r)(P_{d_1|r} + P_{d_2|r} - P_{d_1|r}P_{d_2|r}), \quad (4)$$

where $P_r$ is the FEP of $r$ and $P_{d_k|r}$ is the FEP of $d_k$ on the condition that $r$ can successfully decode $\mathbf{x}_s$. In the sequel, we will optimize the PAFs of the two RNCs to minimize $P_{sys}$ according to the CSI available at the transmitters.

### A. Statistical CSI based Power Allocation Scheme

Due to the statistical symmetry of the channel model, the PAFs are chosen as $\kappa_1 = \kappa_2 = \frac{1}{2}\kappa$ and $\tau_1 = \tau_2 = \frac{1}{2}\tau$. To find the optimal relation between $\kappa$ and $\tau$, we firstly focus on $P_r$. We denote $P_{PE,r}$ as the average pairwise error probability (APEP) of $r$. Since there are in total $2^{2R}$ codewords, we have $P_r = 2^{2R}P_{PE,r}$. By taking expectation with respect to $[g_1, g_2]$, statistical CSI based $P_{PE,r}$ can be deduced as [8], i.e.,

$$P_{PE,r} = \mathcal{E}_{u_{s_1},u_{s_2}}\left\{\frac{\rho^{-1}}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\rho}+\frac{|u_{s_1}|^2+|u_{s_2}|^2}{8\sin^2\theta}\right)^{-1}\mathrm{d}\theta\right\}, \quad (5)$$

where $u_{s_k} = \sqrt{\kappa_k/P}(x_{s_k} - \hat{x}_{s_k})$ is the normalized decoding error of the symbol $x_{s_k}$. When $\rho$ is large, we omit the factor $\frac{1}{\rho}$ inside the integral in (5). Then $P_r$ can be approximated as

$$P_r \approx 2^{2R}\mathcal{E}_{u_{s_1},u_{s_2}}\left\{\frac{2\rho^{-1}}{|u_{s_1}|^2+|u_{s_2}|^2}\right\}. \quad (6)$$

Next, we focus on $P_{d_k|r}$ of the two protocols. In RCNC, joint ML decoding is performed at $d_k$ shown as (3). Since $x_{s_k}$ can achieve more diversity gain than $x_{s_{\bar{k}}}$, then in the high SNR regime, $P_{d_k|r}$ is dominated by the probability of the event that $x_{s_k}$ is successfully decoded but $x_{s_{\bar{k}}}$ is wrongly decoded. So

$$P_{d_k|r}^{RCNC} \approx 2^R\mathcal{E}_{u_{r_{\bar{k}}}}\left\{\frac{2\rho^{-1}}{|u_{r_{\bar{k}}}|^2}+\mathcal{O}(\rho^{-2})\right\}, \quad (7)$$

where $u_{r_{\bar{k}}} = \sqrt{\tau_{\bar{k}}/P}(x_{s_{\bar{k}}} - \hat{x}_{s_{\bar{k}}})$ is the normalized decoding error of the symbol $x_{s_{\bar{k}}}$. While in RGNC, $x_{s_k}$ and $x_r$ are mutually independent and received by $d_k$ in time division channels. So when $\rho$ is large enough, we get

$$P_{d_k|r}^{RGNC} \approx 2^R\mathcal{E}_{u_{s_k},u_r}\left\{\frac{2\rho^{-1}}{|u_{s_k}|^2}+\frac{2\rho^{-1}}{|u_r|^2}\right\}, \quad (8)$$

where $u_r = \sqrt{\tau/P}(x_r - \hat{x}_r)$ is the normalized decoding error of the symbol $x_r$. In the sequel, we give the statistical CSI based power allocation of the two protocols respectively.

*Theorem 1:* When $\rho$ is large enough, the optimal statistical CSI based optimal power allocation is to choose the PAF $\kappa$ as

$$\kappa^c = \frac{\sqrt{2^{R-2}}}{\sqrt{2^{R-2}}+1}\text{for RCNC, } \kappa^g = \frac{\sqrt{2^{R-1}+2}}{\sqrt{2^{R-1}+2}+1}\text{for RGNC} \quad (9)$$

*Proof:* When $\rho$ is large enough, we rewrite (4) as $P_{sys} \approx P_r + P_{d_1|r} + P_{d_2|r}$. Since $\mathcal{E}(|x_{s_k} - \hat{x}_{s_k}|^2) = \mathcal{E}(|x_r - \hat{x}_r|^2) = 4P$, the expectations of the decoding error $\mathcal{E}(|u_{s_k}|^2) = 2\kappa$, $\mathcal{E}(|u_{r_k}|^2) = 2\tau$ and $\mathcal{E}(|u_r|^2) = 4\tau$. Then we approximate the $P_{sys}$ of the two protocols by their upper bounds, i.e.,

$$\begin{aligned} P_{sys}^{RCNC} &\approx 2^{2R}\frac{2\rho^{-1}}{4\kappa} + 2\cdot 2^R\frac{2\rho^{-1}}{2\tau} \\ &= 2^R\rho^{-1}\left(\frac{2^{R-1}}{\kappa}+\frac{2}{\tau}\right). \end{aligned} \quad (10)$$

So the optimal power allocation of RCNC can be worked out by minimizing $\left(\frac{2^R}{2\kappa}+\frac{2}{\tau}\right)$ subject to the power constraint $\kappa + \tau = 1$. On the other hand, in RGNC protocol, we have

$$\begin{aligned} P_{sys}^{RGNC} &\approx 2^{2R}\frac{2\rho^{-1}}{4\kappa} + 2\cdot\left(2^R\frac{2\rho^{-1}}{2\kappa}+2^R\frac{2\rho^{-1}}{4\tau}\right) \\ &= 2^R\rho^{-1}\left(\frac{2^{R-1}+2}{\kappa}+\frac{1}{\tau}\right). \end{aligned} \quad (11)$$

By minimizing $\left(\frac{2^{R-1}+2}{\kappa}+\frac{1}{\tau}\right)$ subject to the power constraint $\kappa+\tau = 1$, we get the optimal power allocation of RGNC. ∎

### B. Instantaneous CSI based Power Allocation

If instantaneous CSI is available at all transmitters, PAFs can be further optimized. In the first time slot, we focus on guaranteeing the quality of both $s \to r$ channels to minimize $P_r$, which is a multi-access channel model. According to [9], we suppose that each source splits its power into $M$ pieces, i.e., $2\kappa_k\rho = M\triangle\rho_k$. Two sources alternatively pour one piece of their power into the channels and gain the rate growth $\triangle R(s_k^m)$ in the $m$-th round. Let $\triangle\rho_k \to 0$. Then we have $\triangle R(s_k^m) = \frac{1}{2}|g_k|^2\triangle\rho_k\eta_m$, where $\eta_m = 1/(1+m\sum_{j=1}^2|g_j|^2\triangle\rho_j)$. When joint ML decoding is performed at $r$, $\triangle\rho_{\bar{k}}$ can be replaced by $\frac{\kappa_{\bar{k}}}{\kappa_k}\triangle\rho_k$, i.e.,

$$\begin{aligned} I(s_k;r|g_1,g_2) &= \int_0^{2\kappa_k\rho}\frac{\frac{1}{2}|g_k|^2\,\mathrm{d}\rho_k}{1+(|g_k|^2+\frac{\kappa_{\bar{k}}}{\kappa_k}|g_{\bar{k}}|^2)\rho_k} \\ &= \frac{\kappa_k|g_k|^2}{\kappa_k|g_k|^2+\kappa_{\bar{k}}|g_{\bar{k}}|^2}I(s_k,s_{\bar{k}};r|g_1,g_2). \end{aligned} \quad (12)$$

Let $I(s_k;r|g_1,g_2) = I(s_{\bar{k}};r|g_1,g_2)$ to guarantee the quality of the worse channel. Then we the power allocation between the two sources as $\kappa_k = \kappa|g_{\bar{k}}|^2/(|g_k|^2+|g_{\bar{k}}|^2)$. Moreover, the phase of each $s \to r$ channel is pre-equalized to ensure the coherent superposition of the two signals. Then we focus on the instantaneous CSI based power allocation in RCNC. Due to space limitations, the discussion on RGNC is omitted.

*Theorem 2:* A suboptimal instantaneous CSI based power allocation for RCNC is to choose the PAFs as

$$\kappa^c = \frac{\sqrt{\eta 2^{R-1}}}{\sqrt{\eta 2^{R-1}}+1},\ \tau^c = \frac{1}{\sqrt{\eta 2^{R-1}}+1},\ \tau_k^c = \frac{\tau|h_k|}{|h_k|+|h_{\bar{k}}|}, \quad (13)$$

where $\eta = \frac{|h_1h_2|^2(|g_1|^2+|g_2|^2)}{|g_1g_2|^2(|h_1|+|h_2|)^2}$.

*Proof:* Since the instantaneous CSI based SFEP can not be exactly worked out, we give a suboptimal method by replacing the statistical SNR in (10) with the instantaneous SNR. Then the suboptimal power allocation is to minimize $\left(\frac{2^{R-1}}{\kappa|g|^2}+\frac{1}{\tau_1|h_2|^2}+\frac{1}{\tau_2|h_1|^2}\right)$ subject to the power constraint

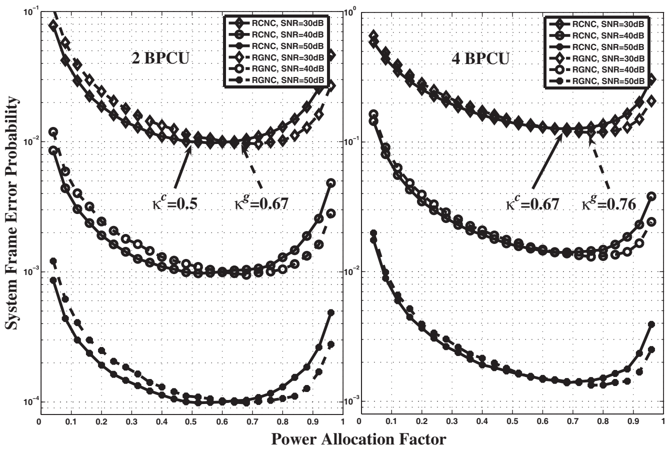


Fig. 2. SFEP with different statistical CSI based PAS for RCNC and RGNC protocols respectively. The horizontal axis represents the PAF $\kappa$.

$\kappa+\tau_1+\tau_2=1$, where $|g|^2=|g_1g_2|^2/(|g_1|^2+|g_2|^2)$. Then we complete the proof. ■

## IV. Analysis and Numerical Results

Consider the conventional scheme without network coding where all signals are transmitted in time division (TD) channels. The transmission of $\mathbf{x}_s$ should take $4$ time slots and thus consumes more power and time slots than that of network coding schemes. So network coding schemes outperform the conventional scheme in terms of system throughput. This issue has been thoroughly investigated in previous works [2]-[6].

In our Monte-Carlo simulations, decoding algorithm and system model are both selected as that in section II. Each SFEP value is simulated by $10^6$ $i.i.d$ frames. Fig. 2 shows the SFEP curves with different values of PAF $\kappa$ where statistical CSI (SCSI) based power allocation of the two protocols are considered. Since the optimal power allocation given by *Theorem 1* are related to $R$, we consider two QAM modulation schemes, i.e., $2$ bit per-channel use (BPCU) and $4$ BPCU. According to (9), in $2$ BPCU scenario, the optimal power allocation for RCNC is to choose $\kappa^c=1/2$ and for RGNC is $\kappa^g=2/3$, while in $4$ BPCU scenario, the optimal power allocation is to choose $\kappa^c=2/3$ and $\kappa^g=\sqrt{10}/(\sqrt{10}+1)\approx 0.76$ for the two protocols respectively. Fig. 2 shows that *Theorem 1* accurately predicts the SCSI based optimal power allocation.

*Theorem 1* also provides a comparison between the two protocols. Note that in multi-access model [5] and unicast model, Galois field network coding outperforms the complex field network coding. However, this is not always true in our multicast system. We compare the performance of the two protocols according to (10) and (11). Let $P_{sys}^{RCNC}=P_{sys}^{RGNC}$, which means that the two protocols have the same system performance. Then we get $\kappa=2/3$ and $\tau=1/3$. If PAF is chosen as $\kappa<2/3$ (or $\kappa>2/3$), we have $P_{sys}^{RCNC}<P_{sys}^{RGNC}$ (or $P_{sys}^{RCNC}>P_{sys}^{RGNC}$). Then RCNC is better (or worse) than RGNC with the performance difference $\Delta P_{sys}=\left|2^R\rho^{-1}(\frac{2}{\kappa}-\frac{1}{\tau})\right|$. Fig. 2 proves our predictions.

Fig. 3 compares the optimal statistical CSI based power allocation scheme (OSPAS) with the instantaneous CSI based power allocation scheme (IPAS) in RCNC. $2$ BPCU and $4$ BPCU are respectively considered. With the instantaneous CSI at each transmitter, IPAS drastically outperforms the OSPAS.

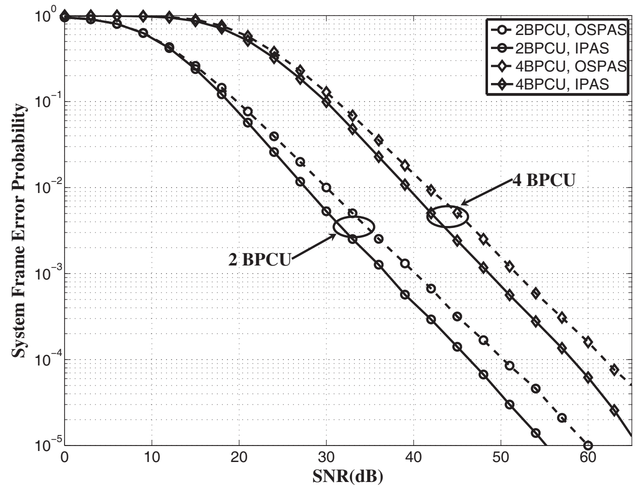


Fig. 3. Optimal statistical CSI based PAS (OSPAS) vs instantaneous CSI based PAS (IPAS) with 2 BPCU and 4 BPCU respectively in RCNC.

## V. Conclusion

In this letter, we analyze the power allocation schemes for RCNC and RGNC protocols in $2-1-2$ multicast system. In the high SNR regime, the optimal statistical CSI based power allocation is proposed by *Theorem 1* in terms of SFEP. According to *Theorem* 1, we also give a comparison of the two RNCs. When instantaneous CSI is available at transmitters, the suboptimal but simple power allocation proposed by *Theorem 2* can further improve the system performance of RCNC.

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