Improved Quantum LDPC Decoding Strategies for the Misidentified Quantum Depolarization Channel

Yixuan Xie, Jun Li, Robert Malaney and Jinhong Yuan

School of Electrical Engineering and Telecommunications
The University of New South Wales, Australia
Email: yixuan.xie@unsw.edu.au, jun.li@njust.edu.cn, r.malaney@unsw.edu.au, j.yuan@unsw.edu.au

***Abstract*—In this work, the importance of the channel mismatch effect in degrading the performance of deployed quantum LDPC codes has been pointed out. We help remedy this situation by proposing new quantum LDPC decoding strategies that can significantly reduce performance degradation by as much as 50%. Our new strategies for the quantum LDPC decoder are based on previous insights from classical LDPC decoders in mismatched channels, where an asymmetry in performance is known as a function of the estimated channel noise. We show how similar asymmetries carry over to the quantum depolarizing channel, and how an estimate of the depolarization flip parameter weighted to larger values leads to significant performance improvement.**

I. Introduction

The existence of quantum error correction codes (QECC) was initially shown by Shor [1], Calderbank [2] and Steane [3], with generalization to stabilizer codes shown by Gottesman [4]. These works, amongst others, outlined the relationships quantum error-correction codes have to classical codes, leading to a pathway for the most successful classical codes, *e.g.* classical LDPC codes, to be readily converted to quantum codes. A more detailed history on the development of QECC can be found elsewhere *e.g.* [5]. Quantum LDPC codes were first proposed in [6], followed by the *bicycle* codes proposed in [7]. More recently, many works attempting to improve quantum LDPC code performance have been published, *e.g.* [8] [9], and [10] based on quasi-cyclic structures.

Low-density parity-check (LDPC) codes were originally proposed by Gallager in early '60s [11], however, LDPC codes remained largely unnoticed until their re-discovery in the mid '90s [12] [13]. Since then many hundreds of papers have been published outlining the near optimal performance of LDPC codes over a wide range of noisy wireless communication channels. In almost all of such previous works it was assumed that the characteristics of the noisy wireless channel was known. However, the reality is that in many cases an exact determination of the wireless channel is unavailable. Indeed, several works have in fact investigated the case where a channel mismatch (or channel misidentification) occurs, which in turn impacts on the performance of the decoder (e.g. [14] [15]).

From the perspective of the work reported on here, the most interesting aspect of such channel mismatch studies is the asymmetry in the LDPC code performance as a function the channel crossover probability for the binary symmetric channel (BSC). In fact, the main focus of the work described here is an investigation of whether such asymmetric LDPC code performance carries over from the classical BSC to quantum LDPC codes operating over the quantum depolarizing channel.

In [16], the impact of channel mismatch effects on the performance of quantum LDPC codes over the quantum depolarizing channel was highlighted, where in practical settings the error performance degrades if no or partial channel information is provided at the decoder side. In this paper, we further investigate the behavior and the robustness of the sum-product (SP) decoder over the quantum depolarizing channel. Interestingly, an asymmetry behavior in performance is observed as a function of the estimated channel flip probability, showing that the performance of a quantum LDPC code would experience a reduced degradation when the channel is overestimated instead of underestimated, provided the overestimated channel knowledge is still within the threshold limit of the code. Based on these observations, a new decoding strategy is proposed that can improve quantum LDPC codes performance by approximately 50%.

In Section II we discuss the behavior of the classical SP decoder under channel mismatch conditions. In Section III we briefly review quantum communications and the *stabilizer* formalism for describing QECCs, and discuss their relationship to classical codes. In Section III we also explore the behavior of a quantum decoder when simulating over a quantum depolarizing channel and show how the decoding strategy we outline here leads to a significant improvement in performance relative to decoders that simply utilize the estimated channel parameter. Lastly, in section IV we draw some conclusions and discuss future works.

II. Behavior of Classical Sum-Product Decoder

It is well known in classical coding that LDPC codes are capacity achievable codes [11] [13], given an optimal decoder. The best algorithm known to decode them is the sum-product algorithm, also known as iterative probabilistic decoding or belief propagation (BP). The performance of sparse-graph codes can be improved if knowledge about the channel is known at the decoder side.

In [14], MacKay *et. al* investigated the sensitivity of Gallager's codes [11] to the assumed noise level when decoded by belief propagation. A useful result therein is that the belief propagation decoder for LDPC codes appears to be robust to channel mismatches because the block error probability is not a very sensitive function of the assumed noise level. In addition, an underestimation of channel characteristics deteriorates the performance more compared to an overestimation of channel characteristics. This behavior is shown in Fig. 1.

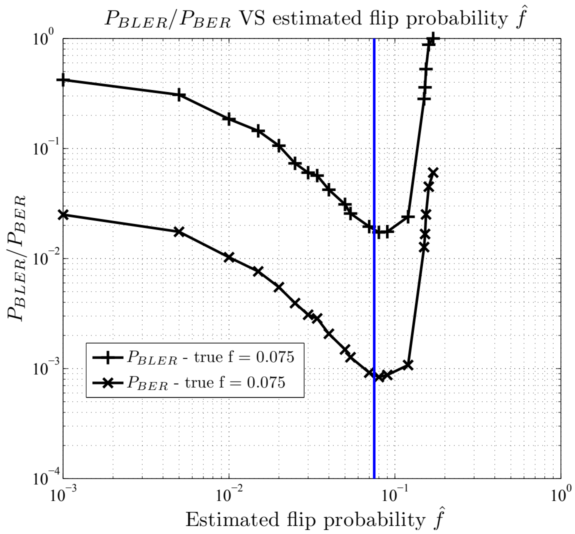

Fig. 1. Probability of block error as a function of estimated flip probability when the true flip probability is fixed.

Our results shown in Fig. 1 are for a rate half code of block length $N = 2040$ over a binary symmetric channel (BSC). The code is a $(3, 6)$ regular LDPC codes which is constructed with the length of the cycle maximized. The plot shows the probability of block and bit error (P_{BLER}/P_{BER}) as a function of assumed flip probability $\hat{f}$ when the true flip probability f is fixed throughout the simulation.

The vertical straight line indicates the true value of the noise level, and the minimum point of the plot is approximately at the intersection between the lines. This infers that an optimal performance of a practical SP decoder can be achieved when the input of decoder is the true noise level. The slope towards the left of the graph is steeper than the slope towards the right, indicating that underestimation of the noise level degrades the performance more so than overestimation does. However, when the estimated noise level is far too large, there is a significant increase in the error probability. Such higher flip probabilities can be thought of as the classical Shannon's limit (in this case, the Shannon's limit for rate 1/2 code is 0.11 computed from $C_{BSC}(f) = 1 - H_2(f)$, where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the binary entropy function), which theoretically represents the threshold (f_{thr}) for the noise level that guarantees reliable transmission at a certain rate.

III. Decoding of Quantum LDPC Codes

Motivated by the decoding asymmetry discussed above for classical LDPC codes, we now wish to explore whether a similar asymmetry in decoding performance is achieved for quantum LDPC codes. As stated below the well-known quantum stabilizer codes can be designed from classical codes. Upon construction of such codes we will then investigate the decoding performance under asymmetrical estimates of the quantum channel parameters. The quantum channel we investigate is the widely adopted depolarization channel, and the BP decoding performed over $GF(4)$.

A. Quantum LDPC Codes

A stabilizer generator S that encodes K qubits in N qubits consists of a set of Pauli operators closed under multiplication, with the property that any two operators in the set *commute*, so that every element of S can be measured simultaneously. Define

$$E \circ F := \prod_{j=1}^{N} E_j \circ F_j \tag{1}$$

as the *commutativity* between two Pauli operators E, F. Then $E_j \circ F_j = +1$ if $E_jF_j = F_jE_j$ and $E_j \circ F_j = -1$ if $E_jF_j = -E_jF_j$. This implies that two Pauli operators either *commute* ($E \circ F = +1$) or *anti-commute* ($E \circ F = -1$).

Consider now a set of error operators $\{E_\alpha\}$ taking a state $|\psi\rangle$ to the corrupted state $E_\alpha |\psi\rangle$. A given error operator either commutes or anti-commutes with each stabilizer S_i (row of the generator S) where $i = 1 \ldots N-K$. If the error operator commutes with S_i then

$$S_i E_\alpha |\psi\rangle = E_\alpha S_i |\psi\rangle = E_\alpha |\psi\rangle \tag{2}$$

and therefore $E_\alpha |\psi\rangle$ is a $+1$ eigenstate of S_i. Similarly, if it anti-commutes with S_i, the eigenstate is -1. The measurement outcome of $E_\alpha |\psi\rangle$ is known as the *syndrome*.

To connect quantum stabilizer codes with classical LDPC codes, a convenient isomorphism between the single qubit Pauli group $\mathcal{P}_1$ generated by Pauli operators $\{I, X, Z, Y = XZ\}$ (the phase factor is ignored under the isomorphism) and the Galois field $GF(4)$ generated by $\{0, 1, \omega, \bar{\omega} = \omega^2\}$ is used. The isomorphism is given by the element identification $I \leftrightarrow 0$, $X \leftrightarrow 1$, $Z \leftrightarrow \omega$ and $Y \leftrightarrow \bar{\omega}$, and the operation identification *multiplication*$\leftrightarrow$ *addition*, *commutativity*$\leftrightarrow$ *trace inner product*. Under this isomorphism, the commutativity between two Pauli operators shown in (1) is equivalent to

$$E \circ F \equiv \mathrm{tr}(\sigma_{\mathrm{E}}\bar{\sigma}_{\mathrm{F}}) \text{ for E, F} \in \mathcal{P}_1 \leftrightarrow \sigma_{\mathrm{E}}, \sigma_{\mathrm{F}} \in \mathrm{GF(4)}, \tag{3}$$

where the trace function $\mathrm{tr(x)} = \mathrm{x} + \mathrm{x}^2$ performs over $GF(2)$. For two vectors $\boldsymbol{\sigma}_{\mathbf{E}} = (\sigma_{E_1}, \sigma_{E_2}, \ldots \sigma_{E_N})$ and $\boldsymbol{\sigma}_{\mathbf{F}} = (\sigma_{F_1}, \sigma_{F_2}, \ldots \sigma_{F_N})$, we have $\boldsymbol{\sigma}_{\mathbf{E}}\bar{\boldsymbol{\sigma}}_{\mathbf{F}} = \sum_i \sigma_{E_i} \times \bar{\sigma}_{F_i}$ as the inner product. Hence, $E \circ F = +1$ if $\mathrm{tr}(\sigma_{\mathrm{E}}\bar{\sigma}_{\mathrm{F}}) = 0$ and $E \circ F = -1$ if $\mathrm{tr}(\sigma_{\mathrm{E}}\bar{\sigma}_{\mathrm{F}}) = 1$.

B. SP Decoding for Quantum LDPC Codes

To decode a quantum LDPC code over the most common quantum channel model, namely the quantum depolarizing channel, it is analogous to decode a classical code over 4-ary symmetric channel [17]. The channel output are diagnosed by the set of $M = N - K$ stabilizer generators and the outcomes $\mathbf{s} \in \{+1, -1\}^M$ (syndrome) is then used as decoder input to estimate possible error occurred. Assuming an initial quantum state representing a codeword, the initial probabilities p_i for the ith qubit of the state undergoing an X, Y or Z error are

$$p_i = \begin{cases} f/3 & for \quad X,\ Y,\ and\ Z \\ 1-f & for \quad\quad I \end{cases}, \tag{4}$$

where f is the flip probability known at the decoder.

The standard BP algorithm operates by sending messages along the edges of the Tanner graph. Let $u_{q_i \to c_j}$ and $u_{c_j \to q_i}$ denote the messages sent from qubit node i to check node j and messages sent from check node j to qubit node i, respectively. Also denote $N(q_i)$ as the neighbors of qubit node i, and $N(c_j)$ as the neighbors of check node j.

To initialize the decoding algorithm, each qubit node sends out a message vector to all its neighbors equal to its initial probability values $[1-f, f/3, f/3, f/3]$ obtained according to equation (4). Upon reception of these messages, each check node sends out a message to its neighboring qubit node given by

$$u_{c_j \to q_i} = \sum_{\{E_{q_{i'}} | E_{q_{i'}} \circ c_j = s_j)\}} \prod_{q_{i'} \in N(c_j) \backslash q_i} u_{q_{i'} \to c_j} \tag{5}$$

where $N\left(c_j\right) \backslash q_i$ denotes all neighbors of check node j except qubit node i, and the summation is over all possible error sequences $E_{q_{i'}}$. Each qubit node then sends out a message to its neighboring checks given by

$$u_{q_i \to c_j} = p_i \prod_{c_{j'} \in N(q_i) \backslash c_j} u_{c_{j'} \to q_i} \tag{6}$$

where $N(q_i) \backslash c_j$ denotes all neighbors of qubit node i except check node j. Equations (5) and (6) operate iteratively until the error operator is correctly decoded or the maximum pre-determined iteration number is reached. The decoder outputs an tentative decision when an error operator $\hat{E} = (\hat{E}_{q_1} \hat{E}_{q_2} \ldots \hat{E}_{q_N})$ has the same syndrome as $\mathbf{s} = (s_1, \ldots s_M)$, where

$$\hat{E}_{q_i} = \arg \max_{E_{q_i} \in \mathcal{P}_1} \left\{ p_i(E_{q_i} \mid \mathbf{s}) \right\}. \tag{7}$$

Note that to effectively compute the check node operation described in (5), FFT-based SPA is used.

C. Behavior of SP Decoder over Depolarizing Channel

We now investigate the dependence of the performance of a quantum LDPC code on the estimated flip probability $\hat{f}$ of a depolarizing channel using the same quantum LDPC code simulated in [16], which is Code A of [8]. In each decoding process, the decoder performed iterative sum-product algorithm until it either found an error operator or reached a maximum number of 200 iterations. The simulation plots herein is the probability of block error (P_{BLER}) as a function of the estimated flip probability $\hat{f}$.

In the simulation, the noise vectors were randomly generated to have weight exactly fN, where N is the block length of the code ($N = 1034$) and f is the true flip probability for the depolarizing channel. The decoder assumed an estimated flip probability $\hat{f}$. We varied the value of $\hat{f}$ while the the true flip probability f is fixed. The results of our simulations are shown in Fig. 2.

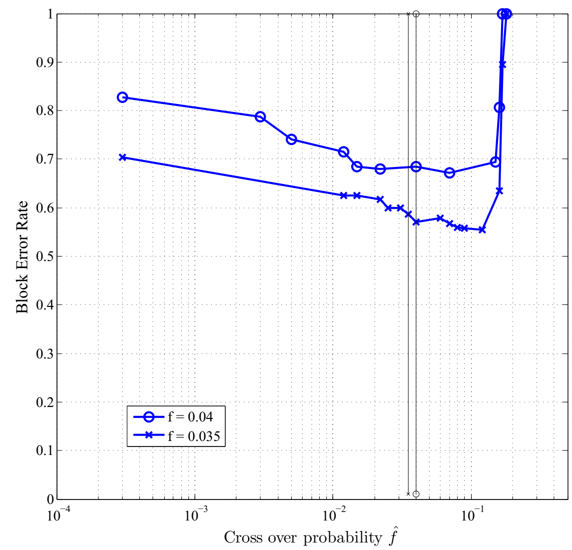

Fig. 2. Probability of block error as a function of estimated flip probability when the true flip probability is fixed.

Similar to the case of classical LDPC codes discussed earlier, we can see from Fig. 2 that optimal performance in the quantum LDPC code can be obtained when the input at the decoder is close to the true flip probability. The trend of the curve in Fig. 2 also shows an overestimate of f is less costly than an underestimate of f, provided that the estimation of channel flip probability, $\hat{f}$, is less than some threshold f^Q_{thr}. For the code shown in Fig. 2, the theoretical threshold is $f^Q_{thr} = 0.1893$ (the capacity of classical 4-ary symmetric channel computed from $C_{4-ary} = 2 - H_2(f) - f \log_2(3)$). In the following section, we show that an improvement in performance of the SP decoder can be achieved if $\hat{f} < f^Q_{thr}$. Note that the P_{BLER} as a function of $\hat{f}$ shown in Fig. 2 is code dependent, which only a small range of P_{BLER} can be shown due to the error correction capability of Code A.

D. Quantum Channel Estimation

The issue of quantum channel identification is of fundamental importance for a range of practical quantum information processing problems (*e.g.* [5]). In the context of LDPC quantum error correction codes, it is normally assumed that the quantum channel is known perfectly in order for the code design to proceed. In reality of course, perfect knowledge of the quantum channel is not available - only some estimate of the channel is available. The quantum depolarization channel of some states can be defined as $\varepsilon(\rho_s) = (1-f)\rho_s + f/3 \sum_{j=1}^{3} \sigma_j \rho_s \sigma_j$, where f is the *true flip probability*.

In what follows we will assume the true value of f is unknown *a priori*, and must first be measured via some channel identification procedure. This estimate of f, which we will refer to as $\hat{f}$, will be used in a decoder in order to measure its performance relative to a decoder in which the true f is utilized.

In general, quantum channel identification proceeds by inputting a known quantum state σ (the probe) into a quantum channel Γ_p that is dependent on some parameter p (in our case $p = f$). By taking some quantum measurements on the output quantum state $\Gamma_p(\sigma)$ which leads to some result R, we then hope to estimate p. The input quantum state may be unentangled, entangled with an ancilla qubit (or qudit), or entangled with another probe. Multiple probes could be used, or the same probe can be recycled (*i.e.* sent through the channel again).

Optimal channel estimation via the use of the quantum Fisher information has been well studied in recent years, particularly in regard to the determination of the parameter f of the depolarizing channel (*e.g.* [18] - [22]). Defining $\rho_f = \Gamma_f(\sigma)$, the quantum Fisher information about f can be written as

$$J(f) = J\left(\rho_f\right) = \mathrm{tr}\left[\rho_f\right] L_f^2,$$

where L_f is the symmetric logarithmic derivative defined implicitly by

$$2\partial_f \rho_f = L_f \rho_f + \rho_f L_f,$$

and where ∂_f signifies partial differential w. r. t. f. With the quantum Fisher information in hand, the quantum Cramer-Rao bound can then be written as

$$\mathrm{mse}\left[\hat{f}\right] \geq (N_m J(f))^{-1}$$

where $\mathrm{mse}\left[\hat{f}\right]$ is the mean square error of the unbiased estimator $\hat{f}$, and N_m is the number of independent quantum measurements.

The performance results in [16] are obtained by randomly estimating a flip probability from a truncated normal distribution at the decoder side, given the mean square error of the unbiased estimation $\hat{f}$.

E. Improved Decoding Strategy

In this section, a numerical approach to improving the performance of the SP decoder is described. The asymmetric behavior of the SP decoder shown in Fig. 2 implies that in the case of channel mismatch, an overestimation of the channel flip probability is more desirable than underestimation.

Consider the case where a decoder can only attain partial channel information by probing the quantum channel using un-entangled or entangled quantum states. Given such partial information we will then weight our estimate of the channel parameter (at the decoder side) to larger values (rather than smaller values) of the estimated flip probability.

For a given true flip probability f, the probability of block error shown in Fig. 2 can be fit approximately by:

$$P_{BLER}^{(f)}(\hat{f}) \approx a + b\hat{f}^3 + c\hat{f}^5 + d\hat{f}^7 + e\sqrt{\hat{f}}\ln(\hat{f}), \qquad (8)$$

where a, b, c, d, e are constants. Assuming our estimator of $\hat{f}$ is centered on the true flip probability (*i.e.* an unbiased estimator), has a variance derived from its quantum Fisher information (*i.e.* an optimal estimator), and has a known probability density function $P(\hat{f})$, we can then make an estimate of what constant should be added to any estimated $\hat{f}$ in order to maximally improve the decoder performance.

Note that, for the case where the qubit probe is in an unentangled state, the quantum Fisher information about f can be shown to be $(N_m J(f))^{-1} = [f(2-f)]$. The average probability of block error for a given f can then be estimated using the equation

$$\tilde{P}_{BLER} = \int_0^{f_{thr}^Q} P(\hat{f}) P_{BLER}(\hat{f}) d\hat{f}. \qquad (9)$$

The performance of the SP decoder can be improved if a factor $\Delta\hat{f}$ is added to the estimated value of $\hat{f}$. That is, $\hat{f} \to \hat{f} + \Delta\hat{f}$. The question then becomes, given some channel what is the optimal $\Delta\hat{f}$ that minimizes the expected probability of error? To answer this, equation (9) is modified to

$$\tilde{P}_{BLER}(\Delta\hat{f}) = \int_0^{f_{thr}^Q} P(\hat{f}) P_{BLER}(\hat{f} + \Delta\hat{f}) d\hat{f}, \qquad (10)$$

The optimal $\Delta\hat{f}$ is then the solution to

$$\frac{\partial}{\partial \Delta\hat{f}} \tilde{P}_{BLER}(\Delta\hat{f}) = 0. \qquad (11)$$

One could repeat this process for a range of true channel flip probabilities, and derive an estimate of the $\Delta\hat{f}$ averaged over the range of true flip probabilities where QECC can be expected to be of relevance, that is

$$\Delta\hat{f}_{avg} = \int_0^{f_{thr}^Q} P(f|\hat{f}) \Delta\hat{f} df. \qquad (12)$$

For the same code (Code A) as that used in Fig. 2, assume a uniform distribution for $P(f|\hat{f})$, and taking $N_m = 1$ in the Fisher information, we found that value of $\Delta\hat{f}_{avg}$ to be very weakly dependent on f (see Table I). This means that simply adding to each estimated $\hat{f}$ the additional factor $\Delta\hat{f}_{avg}$ led to substantial performance improvement. The magnitude of this improvement can be seen in Fig. 3. In this figure $\Delta\hat{f}_{avg} \approx 0.01422$ is applied at the the decoder to provide the improved error correction (shown are the fraction of blocks in error P_{BLER}), denoted as 'CodeA $-\ \hat{f} + \Delta\hat{f}_{avg}$'. The notation 'CodeA $-\ \hat{f}$' in this figure is for the case where the input to the SP decoder is $\hat{f}$ only, whereas the notation 'CodeA $-\ f$' is for the case where the input to the decoder is the true flip probability f. As can be seen improvements of up to $\sim 50\%$ can found from the new strategy ($\hat{f} + \Delta\hat{f}_{avg}$), relative to the case of just utilizing the estimated $\hat{f}$. Similar results to those shown were found for other codes investigated, although the factor to be added was found to be a function of the code. For example, in another code investigated (Code B using Construction method III of [8] code length $N = 2068$) a $\Delta\hat{f}_{avg} \approx 0.00365$ was found to be better and the performance improvement is up to $\sim 30\%$ relative to the case of utilizing the estimate $\hat{f}$ (the corresponding optimal $\Delta\hat{f}$ for each different true flip probability f for Code B is also listed in TABLE I and see also Fig. 3 for simulation improvement). Of course, improved channel estimation also alters the details of our analysis, with more accurate measurements (*e.g.* a higher number of measurements N_m of the channel) leading to smaller $\Delta\hat{f}_{avg}$, and smaller improvements in performance.

TABLE I
OPTIMAL $\Delta\hat{f}$ FOR DIFFERENT f.

Code A		Code B	
f	$\Delta\hat{f}$	f	$\Delta\hat{f}$
0.04	0.02638	0.05	0.00554
0.03	0.01659	0.04	0.00471
0.02	0.01292	0.03	0.00397
0.01	0.00097	0.02	0.00268
		0.01	0.00134

Finally, it is perhaps worth illustrating how the use of optimal $\Delta\hat{f}$ for each f (denoted as 'CodeA$-\hat{f}+\Delta\hat{f}$' in Fig. 3), rather than $\Delta\hat{f}_{avg}$ for every f, impact the results. From Fig. 3 we can see that if the optimal $\Delta\hat{f}$ for each true f is applied for $f > 0.025$, the error performance is better compared to the case of using $\Delta\hat{f}_{avg}$ for every f (see the magnified portion in Fig 3). This is true since $\Delta\hat{f}_{avg}$ provides excess weight for small f and less weight for large f.

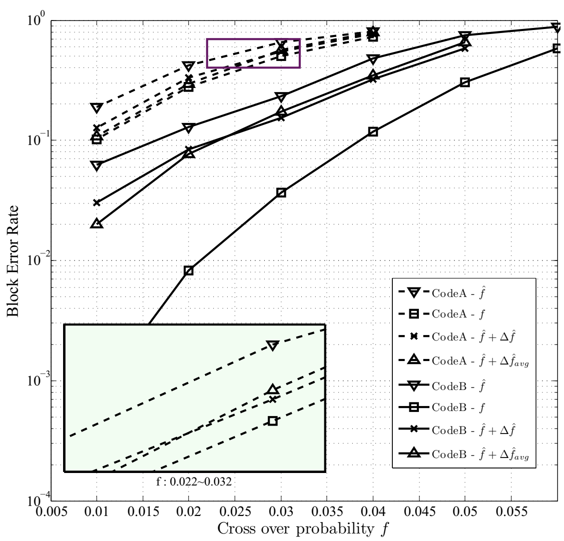

Fig. 3. Comparison of block error rate of Codes A and B.

IV. CONCLUSION

In this work we have investigated possible improvements in the decoding strategies of quantum LDPC decoders in the quantum depolarization channel. The importance of the channel mismatch effect in determining the performance of quantum LDPC codes has very recently been shown to lead to a degradation in the qubit error performance. In this work we have illustrated how such a performance gap in the qubit error performance can be substantially reduced. The new strategies for quantum LDPC decoding we provided here are based on previous insights from classical LDPC decoders in mismatched channels, where an asymmetry in performance is known as a function of the estimated bit-flip probability. We first showed how similar asymmetries carry over to the quantum depolarizing channel. We then showed that when a weighted estimate of the depolarization flip parameter to larger values is assumed, performance improvement by as much as 50% was found.

The work outlined here will be of practical importance when large-scale quantum networks are built, and sophisticated quantum error correction codes are deployed in order to maintain the entanglement between the distributed entangled qubit pairs that underpins these emerging networks. The strategies described here will ultimately manifest themselves in an improved performance of entanglement-based quantum key distribution, or any other entanglement-based quantum communication application, deployed over such future networks.

REFERENCES

[1] P. W. Shor, "Scheme for reducing decoherence in quantum memory," *Phys. Rev. A,* vol. 52, pp. 2493 – 2496, 1995.
[2] A. R. Calderbank and P. W. Shor, "Good quantum error-correcting codes exist," *Phys. Rev. A*, vol. 54, pp. 1098 – 1105, 1996.
[3] A. M. Steane, "Error Correcting Codes in Quantum Theory," *Phys. Rev. Lett.*, vol. 77, pp. 793 – 797, 1996.
[4] D. Gottesman, "Class of quantum error-correcting codes saturating the quantum Hamming bound," *Phys. Rev. A*, vol. 54, pp. 1862 – 1868, 1996.
[5] M. Nielsen and I. Chuang, "Quantum computation and quantum information," *Cambridge Uni. Press*, New York, 2000.
[6] M. S. Postol, "A proposed quantum low density parity check code," *arXiv:quant-ph/0108131v1*, 2001.
[7] D. MacKay, G. Mitchison, and P. McFadden, "Sparse-graph codes for quantum error correction," *IEEE Trans. on Inf. Theory*, vol. 50, pp. 2315 – 2330, 2004.
[8] P. Tan and J. Li, "Efficient quantum stabilizer codes: LDPC and LDPC-convolutional constructions," *IEEE Trans. on Inf. Theory*, vol. 56, pp. 476 – 491, 2010.
[9] M. Hagiwara, K. Kasai, H. Imai, and K. Sakaniwa, "Spatially coupled quasi-cyclic quantum LDPC codes," *Proc. IEEE Int. Symp. Inf. Theory*, pp. 638 – 642, 2011.
[10] K. Kasai, M. Hagiwara, H. Imai, and K. Sakaniwa, "Quantum Error Correction beyond the Bounded Distance Decoding Limit", *IEEE Trans. Inf. Theory*, vol. 58, no. 2, pp. 1223 – 1230, 2012.
[11] R. G. Gallager, "Low-density parity-check codes," *PhD thesis*, MIT, 1963.
[12] D. J. C. MacKay and R. M. Neal, "Near Shannon limit performance of low density parity check codes" *Electronic Lett.*, vol. 32, pp. 1645 – 1646, 1996.
[13] D. MacKay, "Good error-correcting codes based on very sparse matrices," *IEEE Trans. on Inf Theory*, vol. 45, pp. 399 – 431, 1999.
[14] D. MacKay, C. Hesketh, "Performance of Low Density Parity Check Codes as a Function of Actual and Assumed Noise Levels", *Electronic Notes in Theoretical Computer Science*, vol. 74, 2003.
[15] Y. Fujiwara "Instantaneous Quantum Channel Estimation during Quantum Information Processing," *arxiv.org/abs/1405.6267*, 2014.
[16] Y. Xie, J. Li, R. Malaney, J. Yuan, "Channel Identification and its Impact on Quantum LDPC Code Performance", *IEEE Proc. on Australian Comm. Theory Workshop*, pp. 140 – 144, 2012.
[17] D. Poulin and Y. Chung, "On the iterative decoding of sparse quantum codes," *Quantum Inf. Computation*, vol. 8, pp. 987 – 1000, 2008.
[18] A. Fujiwara, "Quantum channel identification problem," *Phys. Rev. A*, vol. 63, p. 042304, 2001.
[19] M. Sasaki, M. Ban, and S. M. Barnett, "Optimal parameter estimation of a depolarizing channel," *Phys. Rev. A*, vol. 66, p. 022308, 2002.
[20] A. Fujiwara and H. Imai, "Quantum parameter estimation of a generalized Pauli channel," *Journal of Phys. A: Mathematical and General*, vol. 36, no. 29, pp. 8093 – 8103, 2003.
[21] M. R. Frey, A. L. Miller, L. K. Mentch, and J. Graham, "Score operators of a qubit with applications," *Quantum Info. Processing*, vol. 9, pp. 629 - 641, 2010.
[22] M. R. Frey, D. Collins, and K. Gerlach, "Probing the qudit depolarizing channel," *Journal of Phys A: Mathematical and Theoretical*, vol. 44, no. 20, p. 205306, 2011.