Novel Nested Convolutional Lattice Codes for Multi-Way Relaying Systems over Fading Channels

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***Abstract*—In this paper, we focus on the realization of multiple interpretations (MI) in multi-way relay channels (MWRC) with fading, where multiple sources communicate with each other with the help of a relay. We first propose a novel nested convolutional lattice codes (NCLC) over the finite field, which can achieve the MI for each source in two time slots. Then we derive a theoretical upper bound for the codeword error rate (WER) of the NCLC. We further optimize our NCLC by developing a code design criterion which minimizes the derived WER. In simulations, we construct a specific NCLC based on our code design criterion. Simulation results show that our code can realize MI for each source in two time slots, and validate the derived upper bound in the high normalized signal-to-effective-noise ratio ($\mathrm{SENR_{norm}}$) region.**

I. Introduction

Future communications are envisioned to support high data rate and large coverage. As a coding approach to enhance the high data rate, network coding [1], [2] has attracted great attentions. A noticeable network coding model is the multi-way relay channel (MWRC) [3]–[5], where all the users exchange full information via a single relay node and there are no direct connections among the users. A common application scenario of the MWRC is the satellite transmission, where all the stations exchange their information via a satellite.

An important concept in the MWRC is multiple interpretations (MI) [6], namely, each user can decode all the other users' information from a network coded packet sent by the relay node. In [6], the authors propose a novel nested code to achieve the goal of MI. The basic idea in [6] is that different packets encoded with different linearly independent generators are combined at a relay node and then forwarded to different destination nodes. Authors in [7] jointly design the convolutional codes with the nested code, and propose a code design criterion to optimize the code profiles. However, the aforementioned coding schemes are limited by the binary codes, and only consider the transmission in orthogonal channels. Thus, the spectrum efficiency in these schemes are relatively low.

Recently, much significant work has been done to construct nested lattice codes to improve the spectrum efficiency in the multi-user relay networks, e.g., compute-and-forward [8] proposed by Nazer and Gasper. In [9], Erez and Zamir show that nested lattice codes can achieve the capacity of the additive white Gaussian noise (AWGN) channel. In [10], a general algebraic framework, called lattice network coding, is developed based on the physical-layer network coding (PNC) [11] schemes. These works provide the idea of simultaneous transmissions from the source nodes to the relay nodes with multi-user interference, and thus achieve a high spectrum efficiency. However, these works cannot achieve MI with one relay node due to the constraints of the code structures.

In this paper, we are interested in achieving MI with high spectrum efficiency in MWRC with fading. Specifically, we consider the transmissions in two time slots, i.e., all the users transmit their information to the relay simultaneously (with multi-user interference) in the first time slot, and the relay broadcasts the network coded information to all the users in the second time slot. Our contributions in the paper are as follows. We first propose a novel nested convolutional lattice codes (NCLC) over the finite field, which can achieve MI for each source in two time slots. Then we derive a theoretical upper bound for the codeword error rate (WER) of the NCLC. Furthermore, we optimize our NCLC by developing a code design criterion which minimizes the derived WER. In simulations, we construct a specific NCLC based on our code design criterion. Simulation results show that our code can realize MI for each source in two time slots, and validate the derived upper bound in the high normalized signal-to-effective-noise ratio ($\mathrm{SENR_{norm}}$) region.

II. System Model and Preliminaries

In this section, we discuss the system model of a multi-way relay channel, the algebraic principle of nested convolutional codes over the finite field, and the properties of lattice network coding.

A. System Model

Consider a network coding group, which consists of one relay node and a set of transceiver nodes, as shown in Fig. 1, where r denotes the relay node and $[s_1, s_2, \cdots, s_L]$ denote the transceiver nodes. All transmitters are receivers as well. The simplified coding process of the proposed scheme can be described into the following steps:

1) All the nodes except the relay node generate different information messages and encode the messages with mutually linearly independent generators.
2) Each transmitter maps the coded message onto an element of a lattice.
3) The lattice codewords from different source nodes are transmitted over fading channels simultaneously in one time slot.

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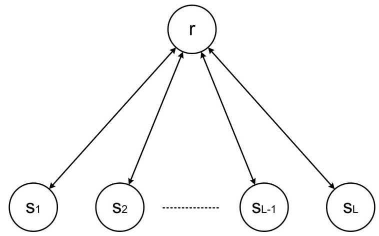


Fig. 1. The system model of the multi-way relay channel.

4) The relay observes the superposition of the lattice codewords, and demodulates it under the constraint of power to obtain the lattice network coded packet.
5) The relay node broadcasts the lattice network coded packet back to the different source nodes through fading channels in the second time slot.
6) Each source's receiver can interpret the desired information from the other sources based on the lattice network coded packet and the prior side information.

More details will be given in the following section.

B. Nested Convolutional Codes over Finite Field

Let $\mathbb{R}$ denote the real field, let $\mathbb{C}$ denote the complex field, let $\mathbb{Z}$ denote the integer field, and let $\oplus$ denote the addition over the finite field. Moreover, let boldface lowercase and uppercase letters denote vectors and matrices respectively.

Let $\mathbb{F}_q$ denote the finite field of size q where q is always assumed to be prime. Let $\mathbf{w}_\ell$ be the message generated independently and uniformly over the $\mathbb{F}_q$ by the source node s_ℓ. Let $\mathbf{G}_\ell$ denote the generator matrix over $\mathbb{F}_q$ at source node s_ℓ, and $\mathbf{G}^T$ denote the transpose of $\mathbf{G}$.

The nested convolutional codes over $\mathbb{F}_q$ are developed based on the nested codes [6] [12], which can be operated over binary to realize MI at multiple destinations [7]. Turn to the case of nested convolutional codes over $\mathbb{F}_q$, we still can observe the similar algebraic property when the coding field of the nested codes is expanded to $\mathbb{F}_q$. Thus, the mathematical operation of nested convolutional codes over $\mathbb{F}_q$ can be expressed by

$$\begin{aligned}\mathbf{w}_1\mathbf{G}_1 \oplus \mathbf{w}_2\mathbf{G}_2 \oplus \cdots \oplus \mathbf{w}_L\mathbf{G}_L \\ =[\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_L]\left[\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_L\right]^T ,\end{aligned} \tag{1}$$

where $\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_L$ are mutually linearly independent.

C. Lattice Network Coding

Let w be a complex number such that the ring of integers $\mathbb{Z}[w] \triangleq \{a + bw | a, b \in \mathbb{Z}\}$ is a principle ideal domain (PID). In this paper, a $\mathbb{Z}[w]$-lattice is defined as [10],

$$\Lambda = \left\{\boldsymbol{\lambda} = \mathbf{G}_\Lambda \mathbf{c} : \mathbf{c} \in \mathbb{Z}[i]^N\right\} , \tag{2}$$

where a lattice generator matrix $\mathbf{G}_\Lambda \in \mathbb{C}^{N\times N}$ is full rank, and N is the number of dimensions.

A coarse lattice Λ' is a sublattice of a fine lattice Λ, i.e., $\Lambda' \subset \Lambda$. For the lattice network codes, the message space $\mathcal{W}$ is $\mathcal{W} = \Lambda/\Lambda'$, which can also be regarded as a module. The message rate for each user is defined as

$$R \triangleq \frac{1}{n}\log_2 |\mathcal{W}| . \tag{3}$$

A map $\mathcal{Q}_\Lambda : \mathbb{C}^N \to \Lambda$ is defined as a nearest-lattice-point (NLP) quantizer, which sends a point $\mathbf{x} \in \mathbb{C}^N$ to a nearest lattice point in Euclidean distance, i.e.,

$$\mathcal{Q}_\Lambda(\mathbf{x}) \triangleq \arg\min_{\boldsymbol{\lambda}\in\Lambda} \|\mathbf{x} - \boldsymbol{\lambda}\| . \tag{4}$$

Let $[\mathbf{s}] \bmod \Lambda$ denote the quantization error of $\mathbf{s} \in \mathbb{C}^N$ with respect to the lattice Λ,

$$[\mathbf{s}] \bmod \Lambda = \mathbf{s} - \mathcal{Q}_\Lambda(\mathbf{s}) . \tag{5}$$

Let $\mathcal{V}$ denote the fundamental Voronoi region of a lattice, which is the set of all points in $\mathbb{C}^N$ that are closest to the zero vector,

$$\mathcal{V} = \{\mathbf{s} : \mathcal{Q}_\Lambda(\mathbf{s}) = 0\} . \tag{6}$$

Let $\psi(\mathbf{w})$ denote a map labeling the message to the points of Λ, similarly, $\psi^{-1}(\boldsymbol{\lambda})$ signifies the inverse mapping process, i.e.,

$$\boldsymbol{\lambda} = \psi(\mathbf{w}) \ , \text{ and } \ \mathbf{w} = \psi^{-1}(\boldsymbol{\lambda}) , \tag{7}$$

where $\boldsymbol{\lambda} \in \Lambda$ and $\mathbf{w} \in \mathbb{F}_q$.

III. Nested Convolutional Lattice Codes

For the NCLC, we regard the finite field $\mathbb{F}_q$ of the nested convolutional codes as the message space $\mathcal{W}$ of the lattice network codes. Thus, the coded messages $\mathbf{w}_\ell\mathbf{G}_\ell$ are uniformity distributed on an equivalent field with the message space $\mathcal{W}$. Besides, we define a function $\digamma$ to represent the operation over a finite field, for example, $\digamma_q(\mathbf{w})$ denotes the operation over $\mathbb{F}_q$, and $\digamma_{\Lambda'}(\boldsymbol{\lambda})$ denotes the operation over the fundamental Voronoi region of Λ'. Then, we can have,

$$\begin{aligned}\digamma_q(\mathbf{w}) = [\mathbf{w}] \bmod q \ , \ \digamma_{\Lambda'}(\boldsymbol{\lambda}) = [\boldsymbol{\lambda}] \bmod \Lambda' , \\ \text{and } \digamma_{\Lambda'}(\boldsymbol{\lambda}) = \psi\left(\digamma_q(\mathbf{w})\right) \ , \ \digamma_q(\mathbf{w}) = \psi^{-1}\left(\digamma_{\Lambda'}(\boldsymbol{\lambda})\right) .\end{aligned} \tag{8}$$

For the convenience of subsequent proof, each encoder is given a dither vector $\mathbf{d}$, which is generated independently according to a uniform distribution over $\mathcal{V}(\Lambda')$, the fundamental Voronoi region of Λ'. The dither vectors $[\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_L]$ generated at source nodes are made available to the relay and the dither vector $\mathbf{d}_r$ generated at the relay is made available at each source node.

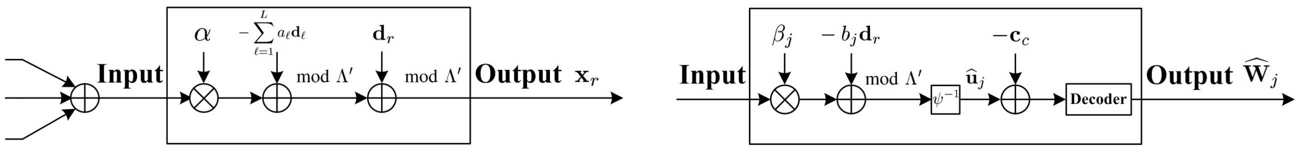
Let $\mathbf{t}_\ell$ denote the coded messages on the fine lattice points, and $\mathbf{x}_\ell$ denote the transmitted signal. Then we have,

$$\mathbf{t}_\ell = \psi(\mathbf{w}_\ell\mathbf{G}_\ell) \text{ and } \mathbf{x}_\ell = [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda' . \tag{9}$$

Each transmitted signal is subject to an average power constraint given by,

$$\frac{1}{n}E\left[\|\mathbf{x}_\ell\|^2\right] \le P . \tag{10}$$

In the following part, we describe the coding process of NCLC in more details.



(a) The lattice network coding process at the relay node.

(b) The decoding process at the receiver of the source node s_j.

Fig. 2. Processes of the NCLC at the relay node and the j_{th} source node s_j.

First, as shown in Fig. 2(a), the relay observes a channel output

$$\mathbf{y}_{sr} = \sum_{\ell=1}^{L} h_\ell \mathbf{x}_\ell + \mathbf{z}_{sr} \ , \tag{11}$$

and the transmitted signal is

$$\begin{aligned} \mathbf{x}_\ell &= [\mathbf{t}_\ell + \mathbf{d}_\ell] \bmod \Lambda' \\ &= [\psi(\mathbf{w}_\ell \mathbf{G}_\ell) + \mathbf{d}_\ell] \bmod \Lambda' \ . \end{aligned} \tag{12}$$

According to the lattice-partition-based compute-and-forward scheme proposed by Nazer and Gastpar [8], we have an optimal scheme at the relay node by choosing some scale factor α, coefficient vector $\mathbf{a} \triangleq (a_1, a_2, \cdots, a_L)$, where $\alpha \in \mathbb{C}$, $\mathbf{a} \in \Lambda$, then we obtain,

$$\begin{aligned} \alpha \mathbf{y}_{sr} &= \sum_{\ell=1}^{L} \alpha h_\ell \mathbf{x}_\ell + \alpha \mathbf{z}_{sr} \\ &= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \underbrace{\sum_{\ell=1}^{L} (\alpha h_\ell - a_\ell)\mathbf{x}_\ell + \alpha \mathbf{z}_{sr}}_{\mathbf{n}} \\ &= \sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \mathbf{n} \ , \end{aligned} \tag{13}$$

where h_ℓ is the channel coefficient of the link between source node s_ℓ and relay r, $\mathbf{z}$ represents the samples of AWGN with zero mean and variance σ^2, and α is derived as [8],

$$\alpha = \frac{P_s \mathbf{h}^H \mathbf{a}}{P_s \|\mathbf{h}\|^2 + N_0} \ , \tag{14}$$

where P_s is the transmission power at each source node, $\mathbf{h}^H$ denotes the Hermitian transpose of $\mathbf{h}$ and N_0 is the average noise power.

To remove dithers under the power constraint, we have,

$$\begin{aligned} &\left[\alpha \mathbf{y}_{sr} - \sum_{\ell=1}^{L} a_\ell \mathbf{d}_\ell\right] \bmod \Lambda' \\ &= \left[\sum_{\ell=1}^{L} a_\ell \mathbf{x}_\ell + \mathbf{n} - \sum_{\ell=1}^{L} a_\ell \mathbf{d}_\ell\right] \bmod \Lambda' \\ &= \left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell + \mathbf{n}\right] \bmod \Lambda' \ , \end{aligned} \tag{15}$$

where $\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell$ and $\mathbf{n}$ are independent.

Then, we have the lattice network coded packet from the relay that should be transmitted back to each source node under the power constraint,

$$\begin{aligned} \mathbf{x}_r &= \left[\left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell + \mathbf{n}\right] \bmod \Lambda' + \mathbf{d}_r\right] \bmod \Lambda' \\ &= \left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell + \mathbf{n} + \mathbf{d}_r\right] \bmod \Lambda' \ . \end{aligned} \tag{16}$$

According to the reference [13], since $\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell$ and $\mathbf{n}$ are independent, we can obtain that $\mathbf{x}_r$ and $\mathbf{n}$ are independent.

Next, as shown in Fig. 2(b), the source node s_j receives the lattice network coded packet from the relay node,

$$\begin{aligned} \mathbf{y}_{rs_j} &= h_j \mathbf{x}_r + \mathbf{z}_{rs_j} \\ &= h_j \left(\left[\sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell + \mathbf{n} + \mathbf{d}_r\right] \bmod \Lambda'\right) + \mathbf{z}_{rs_j} \ . \end{aligned} \tag{17}$$

Similarly, we remove the dithers at the j_{th} receiver node by choosing some scalars β_j and b_j, where $\beta_j \in \mathbb{C}$, $b_j \in \Lambda$,

$$\begin{aligned} &\left[\beta_j \mathbf{y}_{rs_j} - b_j \mathbf{d}_r\right] \bmod \Lambda' \\ &= \left[\beta_j h_j \mathbf{x}_r + \beta_j \mathbf{z}_{rs_j} - b_j \mathbf{d}_r\right] \bmod \Lambda' \\ &= \left[b_j \mathbf{x}_r + (\beta_j h_j - b_j)\mathbf{x}_r + \beta_j \mathbf{z}_{rs_j} - b_j \mathbf{d}_r\right] \bmod \Lambda' \\ &= \left[b_j \sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell + \underbrace{b_j \mathbf{n} + (\beta_j h_j - b_j)\mathbf{x}_r + \beta_j \mathbf{z}_{rs_j}}_{\mathbf{m}_j}\right] \bmod \Lambda' \\ &= \left[b_j \sum_{\ell=1}^{L} a_\ell \mathbf{t}_\ell + \mathbf{m}_j\right] \bmod \Lambda' \ , \end{aligned} \tag{18}$$

where the optimal coefficient scalar b_j from the points of the fine lattice should be chosen to close to the channel coefficient according to [8].

Theorem 1: The optimal scale factor β_j should be chosen to maximize the signal-to-effective-noise ratio (SENR) of the Eq. (18). Given the assumption that the Minimum Mean Square Error (MMSE) decoder is employed at each receiver, β_j can be obtained by

$$\beta_j = \frac{b_j P_r h_j}{P_r |h_j|^2 + N_0} \ , \tag{19}$$

where P_r is the transmission power from the relay.

Proof : Please refer to the Appendix.

Thus, we have the estimate of the desired linearly combination as

$$\begin{aligned}\widehat{\mathbf{u}}_j =&\psi^{-1}\left(\mathcal{Q}_\Lambda\left(\left[b_j\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell+\mathbf{m}_j\right] \mod \Lambda'\right)\right)\\ =&\psi^{-1}\left(\mathcal{Q}_\Lambda\left(\left[\digamma_{\Lambda'}\left(b_j\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell\right)+\mathbf{m}_j\right] \mod \Lambda'\right)\right)\\ =&\psi^{-1}\left(\mathcal{Q}_\Lambda\left(\digamma_{\Lambda'}\left(b_j\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell\right)+\mathbf{m}_j-\mathcal{Q}_{\Lambda'}(\Theta)\right)\right)\\ =&\psi^{-1}\left(\digamma_{\Lambda'}\left(b_j\sum_{\ell=1}^{L}a_\ell\psi(\mathbf{w}_\ell\mathbf{G}_\ell)\right)+\mathcal{Q}_\Lambda(\mathbf{m}_j)-\mathcal{Q}_{\Lambda'}(\Theta)\right)\\ =&\digamma_q\left(p_j\sum_{\ell=1}^{L}q_\ell\mathbf{w}_\ell\mathbf{G}_\ell\right)+\psi^{-1}\left(\mathcal{Q}_\Lambda(\mathbf{m}_j)\right)\\ =&\digamma_q\left(p_j\mathbf{WQG}\right)+\psi^{-1}\left(\mathcal{Q}_\Lambda(\mathbf{m}_j)\right)\ ,\end{aligned} \tag{20}$$

where $\mathbf{W}=[\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_L]$, $\mathbf{Q}=\text{diag}\{q_1,q_2,\cdots,q_L\}$, $\mathbf{G}=[\mathbf{G}_1,\mathbf{G}_2,\cdots,\mathbf{G}_L]^T$, $p_j=\psi^{-1}([b_j] \mod \Lambda')$, $q_\ell=\psi^{-1}([a_\ell] \mod \Lambda')$, $p_j,q_\ell\in\mathbb{F}_q$, and

$$\Theta=\digamma_{\Lambda'}\left(b_j\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell\right)+\mathbf{m}_j\ . \tag{21}$$

Note that the desired linear combination from the lattice network coded packet received at the j_{th} source node s_j can be formulated as

$$\begin{aligned}\mathbf{u}_j&=\digamma_q\left(p_j\sum_{\ell=1}^{L}q_\ell\mathbf{w}_\ell\mathbf{G}_\ell\right)\\&=\digamma_q\left(p_j\mathbf{WQG}\right)\ .\end{aligned} \tag{22}$$

Therefore, it can be observed from Eqs. (20) and (22) that $\widehat{\mathbf{u}}_j=\mathbf{u}_j$ if and only if $\psi^{-1}\left(\mathcal{Q}_\Lambda(\mathbf{m}_j)\right)=0$, or equivalently, $\mathcal{Q}_\Lambda(\mathbf{m}_j)\in\Lambda'$.

Furthermore, each source node can reduce the decoding complexity by canceling the prior known information from the desired linear combination. Let $\mathcal{K}_j$ denote the indices of the information prior known to the j_{th} receiver. Let $\mathbf{c}_u$ denote the collection of unknown information and $\mathbf{c}_c$ be the collection of known information. We have,

$$\begin{aligned}\mathbf{c}_u&=\mathbf{u}_j\ominus\mathbf{c}_c\\&=\digamma_q\left(p_j\sum_{\ell=1}^{L}q_\ell\mathbf{w}_\ell\mathbf{G}_\ell\right)\ominus\digamma_q\left(p_j\sum_{\ell\in\mathcal{K}_j}q_\ell\mathbf{w}_\ell\mathbf{G}_\ell\right)\\&=\digamma_q\left(p_j\sum_{\ell\notin\mathcal{K}_j}q_\ell\mathbf{w}_\ell\mathbf{G}_\ell\right)\ ,\end{aligned} \tag{23}$$

where $\ominus$ denotes the subtraction over $\mathbb{F}_q$.

Given the assumption that all the sources prior know all the assigned generators and can obtain a sequence of coefficients $\mathbf{q}$. Then each receiver node can decode the unknown messages of all the other nodes and realizes MI in two time slots.

IV. Performance Analysis

In this section, we derive a theoretical upper bound for the WER of the NCLC and further optimize our NCLC by developing a code design criterion which minimizes the derived WER.

First, we derive the error probability of $\widehat{\mathbf{u}}_j$ at the j_{th} receiver node,

$$\Pr[\widehat{\mathbf{u}}_j\neq\mathbf{u}_j]=\Pr[\mathcal{Q}_\Lambda(\mathbf{m}_j)\notin\Lambda']\ . \tag{24}$$

With the reference [10], we have

$$\begin{aligned}&\Pr[\mathcal{Q}_\Lambda(\mathbf{m}_j)\notin\Lambda']\leqslant\Pr[\mathbf{m}_j\notin\mathcal{R}_\mathcal{V}(\mathbf{0})]\\ \leqslant&\sum_{\boldsymbol{\lambda}\in\text{Nbr}(\Lambda\backslash\Lambda')}\exp(-\frac{\upsilon\|\boldsymbol{\lambda}\|^2}{2})E\left[\exp(\upsilon\text{Re}\{\boldsymbol{\lambda}^H\mathbf{m}_j\})\right],\forall\upsilon>0\end{aligned} \tag{25}$$

where $\mathcal{R}_\mathcal{V}(\mathbf{0})$ denotes the Voronoi region of $\mathbf{0}$ in the set $\{\Lambda\backslash\Lambda'\}\cup\{\mathbf{0}\}$, and $\text{Nbr}(\Lambda\backslash\Lambda')$ is the set of neighbors of $\mathbf{0}$ in $\{\Lambda\backslash\Lambda'\}$.

Subsequently, refer to Eq. (13) and (18), we have

$$\begin{aligned}&E\left[\exp(\upsilon\text{Re}\{\boldsymbol{\lambda}^H\mathbf{m}_j\})\right]\\ =&E\left[\exp\left(\upsilon\text{Re}\left\{\boldsymbol{\lambda}^H\left(b_j\sum_{\ell=1}^{L}(\alpha h_\ell-a_\ell)\mathbf{x}_\ell+b_j\alpha\mathbf{z}_{sr}\right.\right.\right.\right.\\ &\left.\left.\left.\left.+(\beta_jh_j-b_j)\mathbf{x}_r+\beta_j\mathbf{z}_{rs_j}\right)\right\}\right)\right]\\ =&E\left[\exp\left(\upsilon\text{Re}\{\boldsymbol{\lambda}^H(b_j\alpha\mathbf{z}_{sr}+(\beta_jh_j-b_j)\mathbf{x}_r+\beta_j\mathbf{z}_{rs_j})\}\right)\right]\\ &\prod_\ell E\left[\exp(\upsilon\text{Re}\{\boldsymbol{\lambda}^Hb_j(\alpha h_\ell-a_\ell)\mathbf{x}_\ell\})\right]\\ =&\exp\left(\frac{1}{2}\upsilon^2\|\boldsymbol{\lambda}\|^2\left(|b_j|^2|\alpha|^2\sigma_{sr}^2+P_r|\beta_jh_j-b_j|^2+|\beta_j|^2\sigma_{rs_j}^2\right)\right)\\ &\prod_\ell E\left[\exp(\upsilon\text{Re}\{\boldsymbol{\lambda}^Hb_j(\alpha h_\ell-a_\ell)\mathbf{x}_\ell\})\right]\\ =&\exp\left(\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2N_0\left(|b_j|^2|\alpha|^2+\frac{P_r}{N_0}|\beta_jh_j-b_j|^2+|\beta_j|^2\right)\right)\\ &\prod_\ell E\left[\exp(\upsilon\text{Re}\{\boldsymbol{\lambda}^Hb_j(\alpha h_\ell-a_\ell)\mathbf{x}_\ell\})\right]\ .\end{aligned} \tag{26}$$

Here, we consider the lattice partition as a hypercube, refer to [10], we have

$$E\left[\exp(\text{Re}\{\mathbf{v}^H\mathbf{x}\})\right]\leqslant\exp(\|\mathbf{v}\|^2\delta^2/24)\ , \tag{27}$$

where $\mathbf{x}\in\mathbb{C}^n$ is a complex random vector uniformly distributed over a hypercube $\delta\mathbf{U}\mathcal{H}_n$, $\delta>0$ is a scalar factor, $\mathbf{U}$ is any $n\times n$ unitary matrix, and $\mathcal{H}_n$ is a unit hypercube in $\mathbb{C}^n$ defined by $\mathcal{H}_n=([-1/2,1/2]+i[-1/2,1/2])^n$. Please note that for a hypercube $P = \frac{1}{n}E[\|\mathbf{x}_\ell\|^2] = \delta^2/6$. Thus, we have

$$\begin{aligned}
&E\left[\exp(\upsilon \mathrm{Re}\{\boldsymbol{\lambda}^H \mathbf{m}_j\})\right] \\
=&\exp\left(\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2 N_0\left(|b_j|^2|\alpha|^2+\frac{P_r}{N_0}|\beta_j h_j - b_j|^2+|\beta_j|^2\right)\right) \\
&\prod_\ell E\left[\exp(\upsilon \mathrm{Re}\{\boldsymbol{\lambda}^H b_j(\alpha h_\ell - a_\ell)\mathbf{x}_\ell\})\right] \\
\leq&\exp\left(\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2 N_0\left(|b_j|^2|\alpha|^2+\frac{P_r}{N_0}|\beta_j h_j - b_j|^2+|\beta_j|^2\right)\right) \\
&\prod_\ell \exp\left(\|\upsilon\boldsymbol{\lambda} b_j(\alpha h_\ell - a_\ell)\|^2 P_s/4\right) \\
=&\exp\left(\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2 N_0\left(|b_j|^2|\alpha|^2+\frac{P_r}{N_0}|\beta_j h_j - b_j|^2+|\beta_j|^2\right)\right. \\
&\left.+\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2|b_j|^2\|\alpha\mathbf{h}-\mathbf{a}\|^2 P_s\right) \\
=&\exp\left(\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2 N_0 Q(\alpha,\mathbf{a},\beta_j,b_j)\right),
\end{aligned} \tag{28}$$

where the quantity $Q(\alpha, \mathbf{a}, \beta_j, b_j)$ is defined as

$$\begin{aligned}
&Q(\alpha,\mathbf{a},\beta_j,b_j) \\
=&\frac{P_s}{N_0}|b_j|^2\|\alpha\mathbf{h}-\mathbf{a}\|^2+|b_j|^2|\alpha|^2+\frac{P_r}{N_0}|\beta_j h_j - b_j|^2+|\beta_j|^2 .
\end{aligned} \tag{29}$$

Hence, by choosing $\upsilon = 1/\left(N_0 Q\left(\alpha, \mathbf{a}, \beta_j, b_j\right)\right)$, we obtain

$$\begin{aligned}
&\Pr[\widehat{\mathbf{u}}_j \neq \mathbf{u}_j] = \Pr[\mathcal{Q}_\Lambda(\mathbf{m}_j) \notin \Lambda'] \\
\leqslant&\sum_{\boldsymbol{\lambda}\in \mathrm{Nbr}(\Lambda\setminus\Lambda')} \exp\left(-\frac{\upsilon\|\boldsymbol{\lambda}\|^2}{2}+\frac{1}{4}\upsilon^2\|\boldsymbol{\lambda}\|^2 N_0 Q(\alpha,\mathbf{a},\beta_j,b_j)\right) \\
=&\sum_{\boldsymbol{\lambda}\in \mathrm{Nbr}(\Lambda\setminus\Lambda')} \exp\left(-\frac{\|\boldsymbol{\lambda}\|^2}{4N_0 Q(\alpha,\mathbf{a},\beta_j,b_j)}\right) \\
\approx&K(\Lambda/\Lambda')\exp\left(-\frac{d^2(\Lambda/\Lambda')}{4N_0 Q(\alpha,\mathbf{a},\beta_j,b_j)}\right) \\
=&K(\Lambda/\Lambda')\exp\left(-\frac{3\mathrm{SENR}_{\mathrm{norm}}\gamma_c(\Lambda/\Lambda')}{2}\right),
\end{aligned} \tag{30}$$

where $\gamma_c(\Lambda/\Lambda')$, $d^2(\Lambda/\Lambda')$, and $K(\Lambda/\Lambda')$ denote the nominal coding gain, the squared minimum inter-coset distance, and the number of the nearest neighbors with $d^2(\Lambda/\Lambda')$ of the lattice partition Λ/Λ', respectively. We have $\mathrm{SENR}_{\mathrm{norm}}$ as [14]

$$\mathrm{SENR}_{\mathrm{norm}} = \frac{\mathrm{SENR}}{2^R} = \frac{P}{2^R N_0 Q(\alpha,\mathbf{a},\beta_j,b_j)} . \tag{31}$$

Let $V(\Lambda)$ denote the volume of the Voronoi regions $\mathcal{V}(\Lambda)$. The nominal coding gain can be expressed by [15]

$$\gamma_c(\Lambda/\Lambda') = \frac{d^2(\Lambda/\Lambda')}{V(\Lambda)^{1/N}} , \tag{32}$$

where the number of dimensions N is equivalent to the packet length n.

Next, we consider the WER of $\widehat{\mathbf{W}}_j$ at the j_{th} node. To obtain the error probability of the information matrix $\mathbf{W}_j$ from the desired linear combination $\mathbf{u}_j$, we can regard $\Pr[\widehat{\mathbf{u}}_j \neq \mathbf{u}_j]$ as the crossover probability of a BSC channel. Thus, according to [7], [16], we can obtain,

$$\begin{aligned}
&\Pr[\widehat{\mathbf{W}}_j \neq \mathbf{W}_j] \\
&< \sum_{d=d^2_{free}}^{\infty} a_d\left(4\Pr[\widehat{\mathbf{u}}_j \neq \mathbf{u}_j]\left(1-\Pr[\widehat{\mathbf{u}}_j \neq \mathbf{u}_j]\right)\right)^{\sqrt{d}/2} ,
\end{aligned} \tag{33}$$

where d^2_{free} and a_d denote the minimum squared Euclidian distance and the number of paths at a squared Euclidian distance d from the all-zero path of the convolutional code corresponding to the "stacked" generator matrix $\mathbf{G}$ in (20), respectively.

Since the cancelation process at all the receiver nodes only discards the known information from the estimate of the desired linearly combined packets, it will not change the amount of error and the packets' length. Therefore, the crossover probability is still $\Pr[\widehat{\mathbf{u}}_j \neq \mathbf{u}_j]$ after the cancelation process. In other words, the cancelation process will not change the performance of the derived upper bound.

Finally, we can have the following important design criterion from the Eqs. (29), (30) and (33).

1) The scalars α and β_j should be chosen such that $Q(\alpha, \mathbf{a}, \beta_j, b_j)$ is minimized;
2) The lattice partition Λ/Λ' should be designed such that $K(\Lambda/\Lambda')$ is minimized and $d^2(\Lambda/\Lambda')$ is maximized;
3) The "stacked" convolutional generator matrix should be designed such that d^2_{free} is maximized and a_d is minimized.

V. Numerical and Simulation Results

In this section, we consider a MWRC with three source nodes and one relay node, and assign different linearly independent rate 1/3, memory 1 generators for the all the source nodes. We assume that all the nodes have the same maximum transmission power P, the lattice partition is chosen as a typical Gaussian integer $\mathcal{W} \cong \mathbb{Z}[i]/\delta\mathbb{Z}[i]$, where $\delta = 2+3i$, and the finite field is chosen as $\mathbb{F}_{13}$ where size $q = 13$. Thus, $\mathcal{W} \cong \mathbb{F}_{13}$, the shaping is a rotated hypercube in $\mathbb{C}^N$, and the message rate is $R = \frac{1}{n}\log_2 13$.

Without loss of generality, we select the source node s_3 as an example node. That is, we focus on the realization of MI at s_3 by extracting the information of s_1 and s_2 from the lattice network coded packets sent by the relay. Specifically, s_3 inversely maps each received message from the relay to the corresponding non-binary convolutional codeword in $\mathbb{F}_{13}$, subtracts its own information based on the cancelation process shown in Eq. (23), and decodes the desired information by stacking the generate matrices of s_1 and s_2 as a rate 2/3 generate matrix. Based on the design criterion described in the previous section, we obtain a "stacked" generate matrix with the code rate $2/3$ given as follows:

$$\begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} = \begin{bmatrix} 8+2D & 6+5D & 2+4D \\ 7+12D & 0 & 7D \end{bmatrix}, \tag{34}$$

$$d^2_{free} = 9, \text{ and } a_{d|d=d^2_{free}} = 24. \tag{35}$$

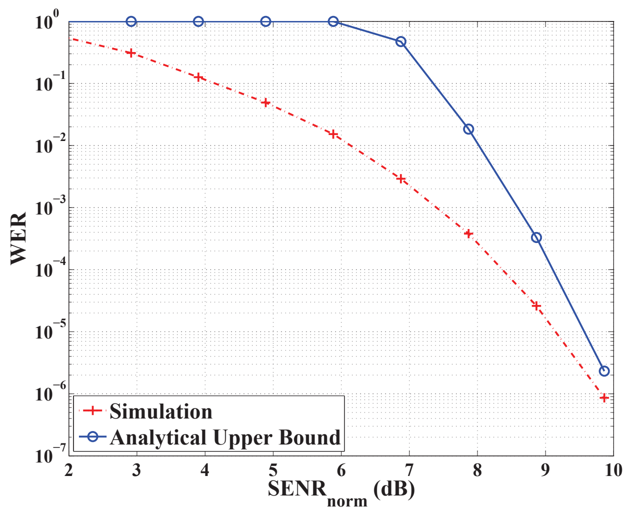


Fig. 3. The Monte-Carlo simulation result and the analytical upper bound.

Next, we investigate the WER of s_3, i.e., $\Pr[\widehat{\mathbf{W}}_3 \neq \mathbf{W}_3]$, based on the code profile given in Eq. (34). We have $\mathbf{W}_3 = [\mathbf{w}_1, \mathbf{w}_2]$, where $\mathbf{W}_3$ is the information matrix at the source node s_3, and is composed by the information $\mathbf{w}_1, \mathbf{w}_2$ from the source nodes s_1 and s_2, respectively. Fig. 3 illustrates the Monte-Carlo simulation result and the analytical upper bound of the WER. The dotted and solid curves denote the simulation result and analytical upper bound, respectively. We can see that the analytical upper bound becomes tight to our Monte-Carlo simulation result as the $\text{SENR}_{\text{norm}}$ increases.

VI. Conclusion

In this paper, we proposed a novel NCLC for MWRC with fading over the finite field, which can achieve MI for each source in two time slots. Particularly, we first presented the detailed coding process of the proposed scheme. Then we derived a theoretical upper bound for the WER of the NCLC and developed a code design criterion by minimizing the derived WER. Finally, in simulations, we constructed a specific NCLC based on our code design criterion. Simulation results show that the NCLC can realize MI for each source in two time slots, and validate the derived upper bound in the high $\text{SENR}_{\text{norm}}$ region.

Appendix
Proof of the Theorem 1

First, we show that maximizing the SENR is equivalent to minimizing $\mathbf{m}_j/b_j$. From Eq. (18), we have

$$\begin{aligned}&\left[b_j\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell+\mathbf{m}_j\right] \mod \Lambda'\\ =&\left[b_j\left(\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell+\frac{\mathbf{m}_j}{b_j}\right)\right] \mod \Lambda'\\ =&b_j\left(\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell+\frac{\mathbf{m}_j}{b_j}\right)-\mathcal{Q}_{\Lambda'}(\Theta)\ ,\end{aligned} \tag{36}$$

where

$$\Theta = b_j\sum_{\ell=1}^{L}a_\ell\mathbf{t}_\ell+\mathbf{m}_j\ . \tag{37}$$

Because $\mathcal{Q}_{\Lambda'}(\Theta)$ is the point on the coarse lattice Λ' and $\psi^{-1}(\mathcal{Q}_{\Lambda'}(\Theta)) = 0$, we can regard $\mathcal{Q}_{\Lambda'}(\Theta)$ as a regular shift of the signal. Hence, β_j should be chosen to minimize $\mathbf{m}_j/b_j$, which is equivalent to maximize the SENR.

It is assumed that the Minimum Mean Square Error (MMSE) detector is employed at each receiver node. Let $f(\beta_j) = E\left[|\mathbf{m}_j/b_j|^2\right]$, from Eqs. (13) (18), we have,

$$\begin{aligned}f(\beta_j) =&\|\alpha\mathbf{h}-\mathbf{a}\|^2P_s+|\alpha|^2N_0\\ &+P_r\left|\frac{\beta_j}{b_j}h_j-1\right|^2+\left|\frac{\beta_j}{b_j}\right|^2N_0\ .\end{aligned} \tag{38}$$

It is apparent that $f(\beta_j)$ is convex. By deriving $f'(\beta_j) = 0$, we have,

$$\beta_j = \frac{b_jP_rh_j}{P_r|h_j|^2+N_0}\ . \tag{39}$$

This completes the proof of Theorem 1. ■

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