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Analysis of Mutual Information Based Soft Forwarding Relays in AWGN Channels

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***Abstract*—In this paper, we analyze the error performance of the mutual information based forwarding (MIF) scheme for a memoryless parallel relay network in additive white Gaussian noise (AWGN) channels. The analytical expression for soft noise variance is first derived. Note that in the literature, the exact soft noise variance could only be evaluated by Monte Carlo simulation due to the lack of its analytical form. The derived soft noise variance expression only relies on the transmit signal-to-noise ratio (SNR), without the need to have the knowledge of actual or estimated information bits. With the expression of the soft noise variance, we derive an approximate bit error rate (BER) expression for a parallel relay network employing MIF scheme. The derived soft noise variance and system BER expressions are shown to be in tight match with Monte Carlo simulation results.**

I. Introduction

For a wireless relay network consisting of several relay nodes, intermediate relay functionality plays a pivotal role in improving the overall system performance. Any wrong decision made at early hops can be propagated to the later ones leading to a poor system performance. Classical memoryless protocols such as amplify-forward (AF) [1] and detect-forward (DetF) [2] suffer from noise amplification and error propagation, respectively. Soft information relaying has been shown to be an effective solution which prevents the error propagation to the destination, and at the same time it preserves the reliability information [3–5]. In [3], a soft forwarding method, namely estimate-forward (EF), has been proposed for memoryless relay networks. It is shown that EF maximizes the generalized signal-to-noise ratio (GSNR) at the destination. A BER optimal solution for the scenario with one relay and no link between source and destination is described in [4]. The approach, however, does not generalize to more than one relay.

Recently, a novel soft forwarding technique based on symbol-wise mutual information, referred to as mutual information based forwarding (MIF), has been proposed for parallel relay networks in AWGN channels [5]. In MIF scheme, each relay node calculates the log-likelihood ratio (LLR) and a corresponding symbol-wise mutual information (SMI) [6] of the received symbols. The sign of the soft decision is determined by the sign of LLR values, and the SMI conditioned on the absolute value of the LLR is used as a reliability measure in generating the soft forwarding symbols. It was shown in [5] that MIF scheme achieves the superior bit error rate (BER) performance compared with other memoryless soft forwarding schemes.

To the best of the authors' knowledge, the error performance of MIF scheme in parallel relay networks has not been reported in the literature. This is mainly due to the lack of appropriate soft noise modeling at the relay and its analytical tractability. In this paper, we aim to address these problems. We start with a single relay channel model. The symmetric property of the soft noise at the relay is first proved. Based on this symmetric property, we derive the analytical expressions for the soft noise variance. In particular, we find that it is possible to calculate the exact soft noise variance based on channel SNR only, and no knowledge of exact or estimated information bits is required. Note that in the previous literature, the soft noise variance could only be determined through Monte Carlo simulation. Then we extend the analysis to parallel relay channels. We demonstrate that the tractability of exact soft noise variance allows to treat a two hop relay channel as a single-hop point-to-point link with an equivalent noise power, which is a function of the soft noise variance of each relay node and the destination noise variance. These results eventually result in the derivation of BER performance of a parallel relay network employing MIF scheme. The soft noise variance and system BER performance, both in analytical forms, are shown to be in tight match with simulation results.

In our analysis, the symmetric property of soft noise plays a significant role. It is noteworthy that many important communication design strategies have been developed exploiting the symmetric property of a random variable. For example, LLR symmetry plays a role in the development of low-density parity-check (LDPC) code optimization technique known as density evolution [7]. The property of 'soft-bit' symmetry was used in finding accurate expressions for extrinsic information transfer (EXIT) function over binary memoryless symmetric channels (BMC) [8]. As another example, 'soft-bit' symmetry property was used in soft turbo channel estimation [9].

II. Soft Information Relay Network

We first consider an elementary relay channel which consists of a source S, a relay R, and a destination D. Later, we extend our analysis for a parallel relay network consisting of K parallel relays as shown in Fig. 1. Let P_S and P_R denote the

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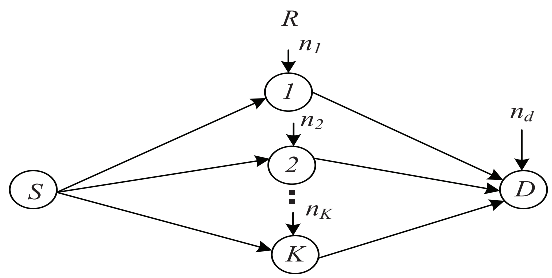

Fig. 1. A parallel relay network.

source and relay symbol energy, respectively. At the source, the binary information streams are mapped into BPSK symbols $x \in \{\pm 1\}$. The channels between all nodes are assumed to be AWGN channels.

In the first phase, the received symbol at the relay is given by

$$r = \sqrt{P_S}x + n, \tag{1}$$

where n is an AWGN with zero mean and variance $\sigma_n^2 = N_0/2$. The SNR is defined as $\frac{E_b}{N_0} = \frac{P_S}{2\sigma_n^2}$. The associated LLR is computed as [10]

$$l = \log \frac{p(r|x=+1)}{p(r|x=-1)} = \frac{2\sqrt{P_S}\,r}{\sigma_n^2} = \frac{2\sqrt{P_S}}{\sigma_n^2}(\sqrt{P_S}x + n). \tag{2}$$

The LLR in (2) can be represented as [10]

$$l = \mu_l x + n_l \tag{3}$$

with $\mu_l = \frac{2\,P_S}{\sigma_n^2}$ and n_l being Gaussian distributed with mean zero and variance

$$\sigma_l^2 = \left(\frac{2\sqrt{P_S}}{\sigma_n^2}\right)^2 \sigma_n^2 = \frac{4\,P_S}{\sigma_n^2} = 8\frac{E_b}{N_0}. \tag{4}$$

Upon receiving r, the relay regenerates a soft symbol $f(l)$, where $f(\cdot)$ denotes the corresponding soft relay function.

In the second phase, the regenerated symbol $f(l)$ is transmitted. The received signal at the destination is

$$y = f(l) + n_d, \tag{5}$$

where n_d is the AWGN with a zero mean and variance σ_n^2.

III. Modeling and analysis of Soft Noise

The soft information in MIF scheme [5] consists of the hard decisions of the symbol estimates and a reliability measure. The reliability measure is determined by SMI computed from the absolute value of the LLR, denoted by λ, at the relay. The MIF based soft information can be written as [5]

$$\tilde{x} = \text{sign}(l)\Theta(\lambda), \tag{6}$$

where $\Theta(\lambda) \in [0, 1]$ denotes the SMI and is given by [6]

$$\Theta(\lambda) = \left(\frac{1}{1+e^{\lambda}}\log_2\frac{2}{1+e^{\lambda}} + \frac{1}{1+e^{-\lambda}}\log_2\frac{2}{1+e^{-\lambda}}\right). \tag{7}$$

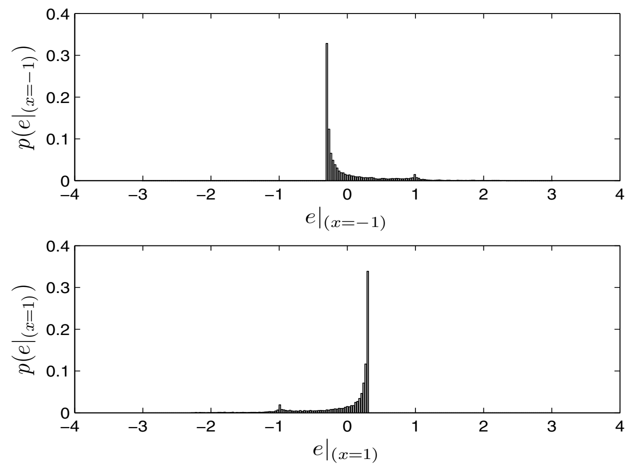

Fig. 2. The normalized histogram of conditional uncorrelated soft noise at $E_b/N_0 = 1$ dB.

For the purpose of performance analysis, we can represent $\tilde{x}$ in terms of information bit and uncorrelated soft noise. Similar to [3], we introduce a scalar coefficient ψ to make the soft noise e uncorrelated with information bit x. To this end, the soft information $\tilde{x}$ can be rewritten as

$$\tilde{x} = \psi(x + e), \tag{8}$$

where e is the uncorrelated soft noise with variance σ_e^2. The parameter ψ is calculated in a way such that x and e become uncorrelated, i.e.,

$$E[xe] = 0. \tag{9}$$

From (8), the soft noise e can be written as

$$e = \tilde{x}/\psi - x. \tag{10}$$

Substituting e from (10) into (9), we have $E[x(\tilde{x}/\psi - x)] = 0$, from which ψ is calculated as

$$\psi = E[x\tilde{x}]. \tag{11}$$

It is clearly seen from (10) and (11) that the knowledge of information bit x is required to calculate the uncorrelated soft noise e, and scalar coefficient ψ. However, in practice it is not possible to know the information bits at the relay. In the following, we will show that the uncorrelated soft noise variance σ_e^2 can be calculated using the symmetric property of soft noise given the channel SNR only. Fig. 2 shows the normalized histogram of conditional uncorrelated soft noise $e|_{(x=\pm 1)}$ in the MIF soft relaying scheme. Now, we state the following Lemma regarding the symmetric property of soft noise e, which would be useful in deriving the analytical uncorrelated soft noise variance.

Now, we state the following lemma.

Lemma 1: The soft noise of MIF based relay network is symmetric, i.e, the soft noise satisfies the following relation.

$$p(e|_{(x=1)}) = p(-e|_{(x=-1)}) \tag{12}$$

Proof: See Appendix A. ■

Theorem 1: The uncorrelated soft noise variance σ_e^2 is given by

$$\sigma_e^2 = \frac{E[\tilde{x}^2|_{(x=1)}]}{\mu_{\tilde{x}}^2|_{(x=1)}} - 1, \tag{13}$$

where

$$E[\tilde{x}^2|_{(x=1)}] = \frac{1}{\sqrt{2\pi}\sigma_l}\int_{-\infty}^{\infty} \tilde{x}^2|_{(x=1)} e^{-\frac{\left(l-\sigma_l^2/2\right)^2}{2\sigma_l^2}} dl, \tag{14}$$

and,

$$\mu_{\tilde{x}}|_{(x=1)} = \frac{1}{\sqrt{2\pi}\sigma_l}\int_{-\infty}^{\infty} \tilde{x}|_{(x=1)} e^{-\frac{\left(l-\sigma_l^2/2\right)^2}{2\sigma_l^2}} dl. \tag{15}$$

Proof: If we substitute (11) into (10), we have

$$e = \frac{\tilde{x}}{E[x\tilde{x}]} - x. \tag{16}$$

If we assume transmitted bits are all '1', i.e, $x = 1$ in (16) and take the expectation on both sides, we can write

$$\mu_{e|_{(x=1)}} = \frac{E[\tilde{x}|_{(x=1)}]}{E[\tilde{x}|_{(x=1)}]} - 1 = 0, \tag{17}$$

where $\mu_{e|_{(x=1)}} = E[e|_{(x=1)}]$.

Again, we can write

$$\sigma_e^2|_{(x=1)} \triangleq E\left[e^2|_{(x=1)}\right] - \mu_e^2|_{(x=1)}. \tag{18}$$

If we substitute $e|_{(x=1)} = \frac{\tilde{x}|_{(x=1)}}{\mu_{\tilde{x}}|_{(x=1)}} - 1$ from (16) and $\mu_{e|_{(x=1)}}$ from (17) into (18), we can write the conditional variance of uncorrelated soft noise as

$$\sigma_e^2|_{(x=1)} = \frac{E[\tilde{x}^2|_{(x=1)}]}{\mu_{\tilde{x}}^2|_{(x=1)}} - 1, \tag{19}$$

where $\mu_{\tilde{x}}|_{(x=1)} = E[\tilde{x}|_{(x=1)}]$. We can also write

$$\begin{aligned}\sigma_e^2|_{(x=1)} &\triangleq E[e^2|_{(x=1)}] - \mu_e^2|_{(x=1)} \\ &= E[e^2] - \mu_e^2|_{(x=1)} \\ &= \sigma_e^2 - \mu_e^2|_{(x=1)},\end{aligned} \tag{20}$$

where the second equality follows from the fact that the conditional and unconditional even order moments of a symmetric random variable are the same. Re-arranging (20), we can write

$$\sigma_e^2 = \sigma_e^2|_{(x=1)} + \mu_e^2|_{(x=1)}. \tag{21}$$

We get Theorem 1 by substituting (17) and (19) into (21). ■

If we substitute (14) and (15) back into (13), then we can see that the expression for uncorrelated soft noise variance σ_e^2 is only a function of σ_l^2. Note from (4) that σ_l^2 is closely related to transmit SNR, denoted by E_b/N_0. Therefore, our derived soft noise variance expression is only determined by transmit SNR. No knowledge of actual or estimated information bits is required

In Fig. 3, the comparison between analytical soft noise variance, given in (13), and simulated one is presented. It is clearly shown that the analytical noise variance closely matches Monte Carlo simulation.

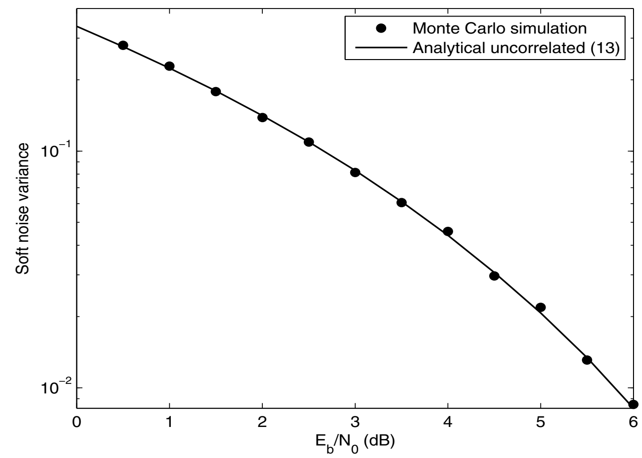

Fig. 3. Soft noise variance for MIF scheme in AWGN channels ($P_S = 1$).

IV. Performance Analysis

In this section we will analyze the bit error rate (BER) of a parallel relay network consisting of one source S, one destination D, and K parallel relays, all of which have a single antenna, as shown in Fig. 1. We assume that the direct link between the source and the destination is not available, which may result from heavy shadowing. There are two phases in the transmission. In the first time slot, the source broadcasts its symbol to the relays. In the second time slot, the K relays transmit simultaneously the regenerated versions of the received symbol to the destination with individual average relay symbol energy $P_{R,i} = P_R/K$. We assume that the relays transmit data through orthogonal channels.

In the first phase, the received symbol at the i-th relay, $i = 1, 2, ..., K$, is

$$r_i = \sqrt{P_S}x + n_i, \tag{22}$$

where n_i is an AWGN with a zero mean and the variance of σ_n^2.

The associated LLR at the i-th relay is computed as

$$l_i = \log\frac{p(r_i|x=+1)}{p(r_i|x=-1)} = \frac{2\sqrt{P_S}\, r_i}{\sigma_n^2}. \tag{23}$$

Upon receiving r_i, the i-th relay regenerates a symbol

$$f(l_i) = \beta_i\tilde{x}_i, \tag{24}$$

where β_i and $\tilde{x}_i$ are the normalization factor and the soft information at the i-th relay, respectively. In the second phase, the relays send the regenerated symbols to the destination. The corresponding received signal from the i-th relay at the destination, denoted by $y_{\text{RD},i}$, can be written as

$$y_{\text{RD},i} = \beta_i\tilde{x}_i + n_{d,i} = \beta_i\psi_i(x + e_i) + n_{d,i}, \tag{25}$$

where e_i is uncorrelated soft noise at the i-th relay, ψ_i is the scalar coefficient to make the soft noise e_i uncorrelated with information bit x, and $n_{d,i}$ is the destination noise with zero

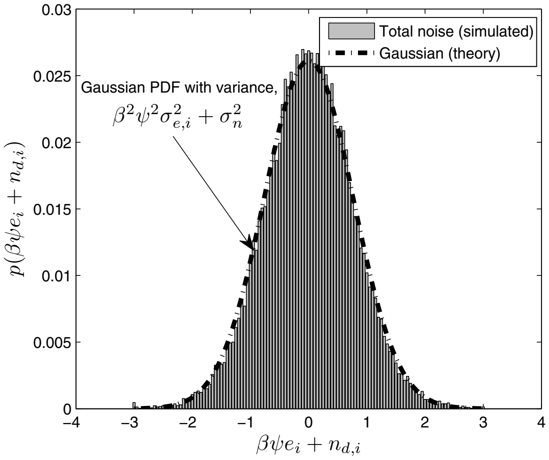

Fig. 4. Comparison between simulated total noise and theoretical Gaussian PDF in AWGN channels at E_b/N_0 =1 dB.

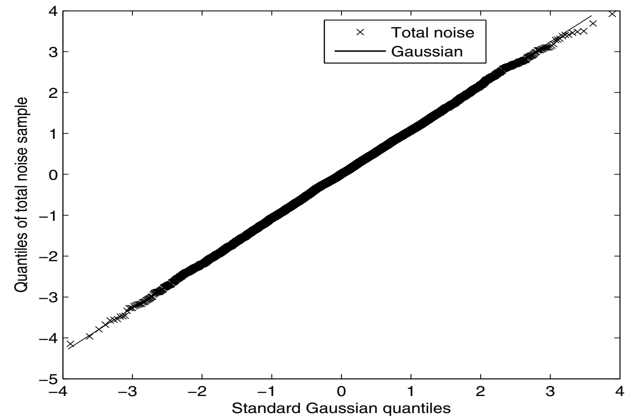

Fig. 5. A Quantile-Quantile (Q-Q) plot for Gaussian noise and total noise at $E_b/N_0 = 1$ dB.

mean and variance $\sigma_n^2 = N_0/2$. The parameter β_i is calculated to meet the relay symbol energy constraint given by

$$E[|f(l_i)|^2] = \beta_i^2\psi_i^2(1+\sigma_{e,i}^2) = P_{R,i}, \tag{26}$$

where $\sigma_{e,i}^2$ is the uncorrelated soft noise variance at the i-th relay. From (26), β_i can be derived as $\beta_i = \sqrt{\frac{P_{R,i}}{\psi_i^2(1+\sigma_{e,i}^2)}}$.

Here, we will examine the distribution of the total noise $\beta_i\psi_i e_i + n_{d,i}$ at the destination. In order to get the distribution of the total noise at the destination, the PDFs of the soft noise and AWGN have to be convolved. Although, the complete distribution of soft noise is not known, we can visualize the resemblance of the soft noise with an impulse-like noise as shown in Fig. 2. As the convolution of a Gaussian distribution with an impulse gives rise to another Gaussian distribution, we conjecture that the total noise at the destination follows a Gaussian distribution.

In Fig. 4, we plot the normalized histogram of combined AWGN and uncorrelated soft noise at the destination or total noise $\beta_i\psi_i e_i + n_{d,i}$. We can see that the histogram of the total noise has a similar shape as Gaussian distribution. In order to further confirm the normality of the total noise at the destination, we resort to a goodness of fit test, known as quantile-quantile (Q-Q) plot construction. In Fig. 5, we plot the quantiles of the theoretical Gaussian distribution against the empirical data points of total noise at the destination. We can see that the quantiles of the total noise at the destination, marked by '$\times$', lies on a 45-degree line, marked by '$-$', representing quantiles of the theoretical Gaussian distribution, which also verifies the approximation of Gaussian distribution of the total noise.

Based on the above discussion, we can approximate the equivalent S-R-D channel as a Gaussian channel. The end-to-end SNR γ_{PAR} from source to destination is given by

$$\gamma_{\mathrm{PAR}} = \sum_{i=1}^{K} \frac{P_R}{P_R\sigma_{e,i}^2 + K\sigma_n^2 + K\sigma_{e,i}^2\sigma_n^2}. \tag{27}$$

As it is assumed that the variance of AWGN n_i at each relay is σ_n^2, we have $\sigma_{e,1}^2 = \sigma_{e,2}^2 = \cdots = \sigma_{e,K}^2 = \sigma_e^2$. As a result, (27) can be written as

$$\gamma_{\mathrm{PAR}} = \frac{KP_R}{P_R\sigma_e^2 + K\sigma_n^2 + K\sigma_e^2\sigma_n^2}. \tag{28}$$

Hence, with the Gaussian approximation of the total noise at the destination, the BER of MIF scheme with K parallel relays is given by

$$P_{b,\mathrm{PAR}} \approx Q\left(\sqrt{2\frac{KP_R}{P_R\sigma_e^2 + K\sigma_n^2(1+\sigma_e^2)}}\right), \tag{29}$$

where $Q(\cdot)$ is the Gaussian-Q function defined as $Q(\alpha) = \frac{1}{\sqrt{2\pi}}\int_\alpha^\infty e^{-t^2/2}dt$. Simulated and analytical BER performance of a MIF based soft information relay network in AWGN channels is presented in Fig. 6 for $K = 1, 2$ and 4. The transmitted power, $P_S = P_{R,i} = 1$ is assumed in the simulation. The 'Analytical' curves are plotted from (29), and they agree with the Monte Carlo simulation results. Note that BER expression for a single relay network can be derived by letting $K = 1$ in (29). Furthermore, we can make the following observations :

A) In a special case of error free source-relay channel, i.e., $\sigma_e^2 = 0$ in (29), $P_b = Q\left(\sqrt{\frac{2P_R}{\sigma_n^2}}\right)$ gives the BER in point to point AWGN channels.

B) In order to get an insight to the effect of relays on BER performance, we can write (28) as

$$\gamma_{\mathrm{PAR}} = \frac{KP_R}{P_R\sigma_e^2 + K\sigma_n^2\left(1+\sigma_e^2\right)} \overset{(a)}{\approx} \frac{KP_R}{P_R\sigma_e^2 + K\sigma_n^2} = \frac{P_R}{\frac{P_R\sigma_e^2}{K} + \sigma_n^2}, \tag{30}$$

where (a) follows from the fact that at high SNR, $\left(1+\sigma_e^2\right) \approx 1$. Compared to the single relay network, i.e., $K = 1$, which introduces an additional noise variance $P_R\sigma_e^2$ to the destination, parallel relay network can effectively reduces the additional noise variance from the relays by a factor K. For

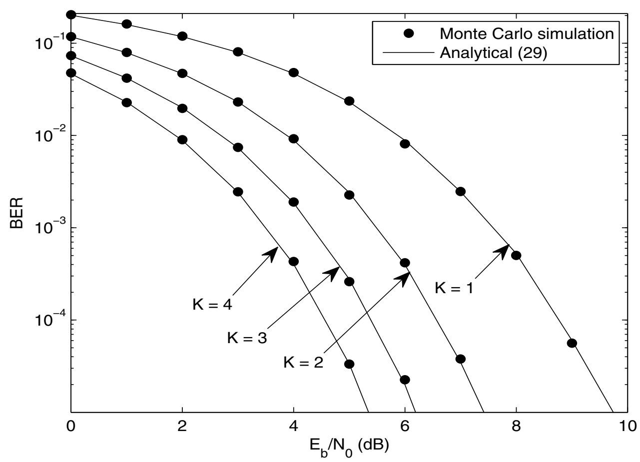

Fig. 6. Analytical and simulated BER performance comparison for soft information relay network in AWGN channels.

asymptotic large network , i.e., $K \to \infty$, the parallel relay network eventually approaches an error free relay network with BER equal to a point-to-point AWGN channel.

V. Conclusion

This paper presented new analytical expressions for soft noise variance. It showed that the soft noise variance can be calculated from its conditional mean and conditional variance. The derived new analytical expressions of soft noise variance featured relaxing the access to the exact or estimated information bits and showed to be calculated by knowing the transmit SNR only. Having soft noise variance analytically, the BER performance of a parallel relay network with MIF scheme was evaluated. All analytical results were confirmed by the Monte Carlo simulations.

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Appendix A
Proof of Lemma 1

From (10), the uncorrelated soft noise e conditioned on $x = 1$ and $x = -1$ can be written as $e|_{(x=1)} = \text{sign}(l|_{(x=1)})\kappa|_{(x=1)} - 1$, and $e|_{(x=-1)} = \text{sign}(l|_{(x=-1)})\kappa|_{(x=-1)} + 1$, respectively, where $\kappa|_{(x=1)} = \frac{\Theta|_{(x=1)}}{\psi|_{(x=1)}}$, and $\kappa|_{(x=-1)} = \frac{\Theta|_{(x=-1)}}{\psi|_{(x=-1)}}$. As both ψ and Θ are not sign preserving functions, we can write $\kappa|_{(x=1)} = \kappa|_{(x=-1)} = \kappa$.

Let $F(e|_{(x=1)})$ denote the cumulative density function (CDF) of $e|_{(x=1)}$. Hence, for an arbitrary value of soft noise ξ, we can write

$$\begin{aligned} F(e|_{(x=1)}) &= P(e|_{(x=1)} \leq \xi) \\ &= P(\text{sign}(l|_{(x=1)})\kappa - 1 \leq \xi) \\ &= P\left(\text{sign}(l|_{(x=1)}) \leq \frac{\xi+1}{\kappa}\right) \\ &\overset{(a)}{=} P\left(\text{sign}(l|_{(x=-1)}) \geq -\frac{\xi+1}{\kappa}\right) \\ &= P\left(\text{sign}(l|_{(x=-1)})\kappa + 1 \geq -\xi\right) \\ &= P\left(e|_{(x=-1)} \geq -\xi\right), \end{aligned} \tag{31}$$

where equality (a) follows from the fact that $p(l)$ is symmetric [7], i.e., $p(l|_{x=+1}) = p(-l|_{x=-1})$, hence $p(\text{sign}(l))$ is also mirror symmetric. The probability density function (PDF) of $e|_{(x=1)}$ is given by

$$\begin{aligned} p(e|_{(x=1)}) &= \frac{dF(e|_{(x=1)})}{d\xi} \\ &\overset{(d)}{=} \frac{d}{d\xi}\left(P\left(e|_{(x=-1)} \geq -\xi\right)\right) \\ &= \frac{d}{d\xi}\left(P\left(-e|_{(x=-1)} \leq \xi\right)\right) \\ &= \frac{d}{d\xi}\left(F(-e|_{(x=-1)})\right) \\ &= p(-e|_{(x=-1)}), \end{aligned} \tag{32}$$

where equality (d) follows from (31).