Design of Distributed Multi-Edge Type LDPC Codes for Multiple Access Relay Channels

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***Abstract*— This paper studies the problem of determining the optimum degree distribution for distributed LDPC codes in multiple-access relay channels. Based on the framework of multi-edge type LDPC codes, we propose a methodology to asymptotically optimize the code's ensemble when different segments within the distributed codeword are transmitted through different channels and experience different SNRs. An average noise threshold is formulated in order to compute the convergence threshold of the distributed LDPC codes under density evolution and also to act as the performance gap between the optimized distributed codes and the theoretical limit. We demonstrate that our distributed LDPC code performs asymptotically to within a fraction of a dB from the theoretical limit.**

I. INTRODUCTION

The concept of network coding proposed by Ahlswede, Cai, Li and Yeung in [1] allows relaying nodes in a network to not only route but also mix the incoming data to destination nodes. Such coding increases the achievable throughput and efficiency of a network compared to conventional routing methods. A network utilizing this concept that has attracted much research interest is the multiple-access relay channel (MARC). The simplest MARC consists of two sources (users), a relay and a destination. The two sources want to send their independent messages to the destination with the help of a common relay that does not have messages of its own to transmit. An example of such a channel model is the cooperative uplink transmission between two mobile terminals and a base station with help from one relay. The fundamental limit of MARC has been widely studied in the literature [2], [3], [4], [5], [6]. In MARC, two main schemes are compress-and-forward (CF) and decode-and-forward (DF). In comparison of these two schemes, the latter outperforms the former when the sources-to-relay links are strong, i.e. the relay is physically closer to the sources as compared to the destination [6].

Earlier works on the application of coding to the MARC under DF scheme can be traced back to the concept of *joint network-channel coding* using Turbo codes [7], [8] and low-density parity-check (LDPC) codes [9], [10]. In the context of LDPC coding for MARC, the work in [9] points out that the joint network-channel coding using distributed LDPC codes performs better than the LDPC coding scheme without network coding. Then, the problem for determining the optimum degree distribution for distributed LDPC codes is investigated in [10] for symmetric MARC, where the channel qualities between both sources to the destination are similar. The symmetric MARC also means that the channels between both sources to the relay are also at the same qualities, i.e. sources can transmit at the same transmission rate using similar LDPC codes, which simplifies the code design problem.

In this paper, we study the problem of determining the optimum degree distribution of distributed LDPC codes for a more general MARC, where we consider an asymmetric time-division MARC (TD-MARC) [1] between both sources to the destination, and also between both sources to the relay, i.e. each source transmits at different transmission rate using different LDPC codes. We extend the general concept of irregular LDPC codes, introduced by multi-edge type (MET) LDPC codes, to the construction of distributed LDPC codes for TD-MARC. A new design technique for distributed LDPC codes is proposed. The rationale for this proposal is the fact that different segments within a distributed codeword have been transmitted through different channels and experience different SNRs. An average noise threshold is formulated to compute the convergence threshold of the distributed LDPC codes under density evolution. We demonstrate that our proposed new codes perform within a fraction of dB away from the theoretical limit.

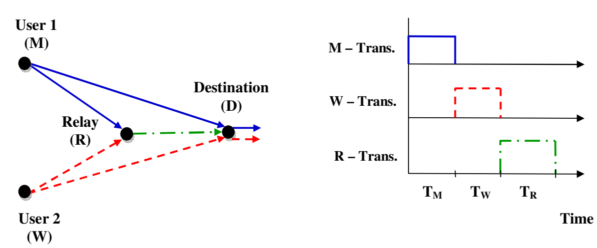

Fig. 1. Time-division multiple-access relay channels.

II. SYSTEM MODEL

A. System description

We consider cooperative communication between two users and one destination, where user 1 (M) and user 2 (W) want to send independent messages to the destination, and in doing so they are aided by one half-duplex relay that does not have

[1]This channel is also known as the orthogonal multi-sources relay channel.

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messages of its own to transmit. This channel can be modeled as the TD-MARC depicted in Fig. 1. The transmitted power for both users and the relay are all set to P. Communication takes place in three phases. The first and second phases T_M and T_W are allocated for user 1 and user 2, respectively, to broadcast their messages to the relay and the destination. Based on the received messages from both users, the relay generates additional messages. The third phase T_B is allocated for the relay to transmit the generated additional messages to the destination. The normalized period of transmission in all phases is $T_M + T_W + T_R = 1$. Within one normalized period of transmission, N symbols are transmitted, i.e. user 1, user 2 and the relay transmit $N_M = T_M \cdot N$, $N_W = T_W \cdot N$ and $N_R = T_R \cdot N$ symbols in each phase, respectively.

Throughout this paper, we adopt the following conventions. M, W, R and D denote user 1, user 2, relay and destination, respectively. For example, the RD channel denotes the relay-to-destination channel. With the above conventions, the received signal at any node in TD-MARC is defined by

$$Y_{pq} = \frac{1}{\sqrt{d_{pq}^{\alpha}}} X_p + Z_{pq} \tag{1}$$

for $p \in \{M, W, R\}$, $q \in \{R, D\}$ and $p \neq q$. Y_{pq} denotes the received signal at node q transmitted from node p, and X_p denotes the transmitted signals from node p. The corresponding distance between node p and node q is d_{pq}. The noise value Z_{pq} at node q is assumed to be Gaussian with zero-mean and unit variance. Thus, the received SNR between two nodes is $\gamma_{pq} = P/(d_{pq}^{\alpha})$, where α denotes the path-loss exponent. In this work, we restricted our attention to the TD-MARC at low signal-to-noise ratio (SNR) for which binary linear codes are near optimum and binary phase shift keying (BPSK) modulation is applied in the transmission. We denote $C(\gamma)$ as the capacity for the binary additive white Gaussian noise (BAWGN) channel.

B. Information-theoretic limits

We briefly explain the achievable DF rate for the TD-MARC with optimum time sharing parameters. Nodes M and W want to transmit statistically independent messages, which are segmented in packets $\mathbf{u}_M$ of length K_M and $\mathbf{u}_W$ of length K_W, respectively. Both messages can be decoded correctly at R and D if the following five inequalities hold [12]

$$K_M \leq N_M \cdot C(\gamma_{MR}) \tag{2}$$

$$K_W \leq N_W \cdot C(\gamma_{WR}) \tag{3}$$

$$K_M \leq N_M \cdot C(\gamma_{MD}) + N_R \cdot C(\gamma_{RD}) \tag{4}$$

$$K_W \leq N_W \cdot C(\gamma_{WD}) + N_R \cdot C(\gamma_{RD}) \tag{5}$$

$$K_M + K_W \leq N_M \cdot C(\gamma_{MD}) + N_W \cdot C(\gamma_{WD}) + N_R \cdot C(\gamma_{RD}) \tag{6}$$

For TD-MARC that satisfies the following conditions: $C(\gamma_{MD}) \leq C(\gamma_{RD})$, $C(\gamma_{WD}) \leq C(\gamma_{RD})$, $C(\gamma_{MD}) \leq C(\gamma_{MR})$ and $C(\gamma_{WD}) \leq C(\gamma_{WR})$, the optimal values of the time-allocation parameters are expressed by [12]

$$T_M^* = C(\gamma_{RD}) \cdot [(1+\varphi\cdot\xi)C(\gamma_{RD}) + (1+\varphi)\cdot C(\gamma_{MR}) - C(\gamma_{MD}) - \varphi\cdot\xi\cdot C(\gamma_{WD})]^{-1} \tag{7}$$

$$T_W^* = T_M^* \cdot \varphi \cdot \xi \tag{8}$$

with $\xi = C(\gamma_{MR}/C(\gamma_{WR}))$ and $\varphi = R_W/R_M$ is the rate ratio between the two users [12]. Here, the two achievable rates for user 1 and user 2 are

$$R_M = \frac{K_M}{N} \quad \text{and} \quad R_W = \frac{K_W}{N},$$

respectively. Note that the two rates can be computed using $R_M = T_M^* \cdot C(\gamma_{MR})$ and $R_W = \varphi \cdot R_M$ [12]. The sum rate of the TD-MARC is $R_T = R_M + R_W = (K_M + K_W)/N$. This achievable DF rate acts as the design rate for our proposed code design in this paper.

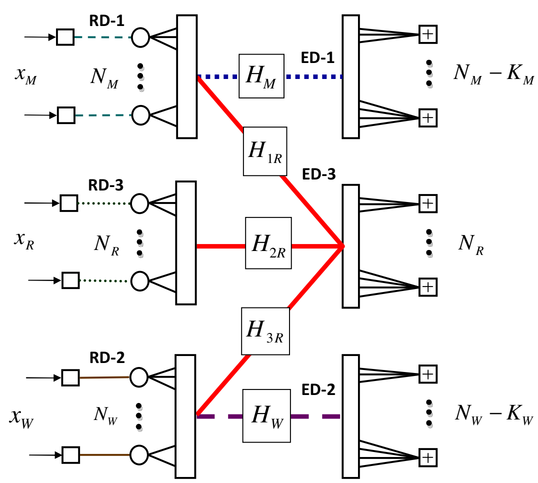

Fig. 2. The distributed LDPC code for TD-MARC. The notation ED-i is the edge degree of type i connecting the variable and check nodes, while RD-j is the received degree connecting the variable nodes to the received symbols transmitted through channel j, where $j \in \{1, 2, 3\}$.

III. LDPC CODING FOR TD-MARC

In this section, we explain how to realize joint network-channel coding (JNCC) for a DF scheme in a TD-MARC using distributed LDPC codes.

In the first phase T_M, M encodes K_M information bits into an LDPC code denoted by vector $\mathbf{X}_M$ of length N_M. To ensure a successful decoding of $\mathbf{X}_M$ at R, the designed rate for $\mathbf{X}_M$ must satisfy

$$R_{MR} = \frac{K_M}{N_M} \leq C(\gamma_{MR}). \tag{9}$$

In the second phase T_W, another LDPC code denoted by vector $\mathbf{X}_W$ of length N_W is generated by W in order to protect and forward K_W information bits to R and D. In this transmission, a successful decoding of $\mathbf{X}_W$ at R is guaranteed if

$$R_{WR} = \frac{K_W}{N_W} \leq C(\gamma_{WR}). \tag{10}$$

After X_M and X_W are decoded correctly, R generates N_R additional parity check bits based on the $N_M + N_W$ decoded symbols (LDPC coded bits). Then, R broadcasts these N_R parity check bits as a vector of $\mathbf{X}_R$ to D in the third phase T_R. The vectors $\mathbf{X}_M$, $\mathbf{X}_W$ and $\mathbf{X}_R$ form a distributed LDPC code, where the relationship between them can be illustrated using the Tanner graph in Fig. 2, and the corresponding parity check matrix for the codes is given by

$$H = \begin{pmatrix} H_M & 0 & 0 \\ 0 & 0 & H_W \\ H_{1R} & H_{2R} & H_{3R} \end{pmatrix}, \quad (11)$$

where H_M and H_W are the sub-parity check matrices correspond to $\mathbf{X}_M$ and $\mathbf{X}_W$, respectively. Moreover, the sub-matrix $[H_{1R}\ H_{2R}\ H_{3R}]$ is used by the relay to generate the additional N_R parity check bits.

Based on observation of $N_M + N_W + N_R$ received signals from the channel output Y_{iD}, $i \in \{M, W, R\}$, the destination will decode the information data of user 1 and user 2 iteratively using a Belief Propagation (BP) algorithm on the overall Tanner graph of the distributed LDPC codes depicted in Fig. 2, i.e. the parity check matrix, H.

IV. DMET-LDPC CODES

The distributed LDPC code is unique in the sense that it has two additional features in the Tanner graph. The first feature is that there is a statistical distinction between the edges in the overall codeword. This is because not all edges can be connected between any two variable and check nodes, e.g. $N_M - K_M$ check nodes can only be linked to N_M variable nodes. This happens because there are two sub-parity check matrices H_M and H_M in the overall parity check matrix H in (11), which are used to generate valid LDPC codewords $\mathbf{X}_M$ and $\mathbf{X}_W$ of user 1 and user 2, respectively. The second feature in the overall Tanner graph is that different segments of the received symbols at D have been transmitted over different channels and experience different SNRs, i.e. the N_M variables go through the MD channel, the N_W variables go through the WD channel and N_R variables go through the RD channel.

The two additional features mentioned above cannot be captured using the conventional framework of irregular LDPC codes for single link channel. As consequence, the codes cannot be optimized under a standard density evolution algorithm [11]. Therefore, we adopt the general concept of irregular LDPC codes introduced by the MET-LDPC codes, where several edge classes (types) can be defined in the LDPC code's structure and every node is characterized by the number of sockets (degrees) in each class. The MET-LDPC codes' ensemble can be specified through the following two multinomials associated to the variable and check nodes [13] [2]

$$\upsilon(\mathbf{r}, \mathbf{x}) = \sum \upsilon_{\mathbf{b},\mathbf{d}} \mathbf{r}^{\mathbf{b}} \mathbf{x}^{\mathbf{d}} \quad \text{and} \quad \mu(\mathbf{x}) = \sum \mu_{\mathbf{d}} \mathbf{x}^{\mathbf{d}}.$$

[2]The MET-LDPC codes are represented from the node-perspective, as opposed to the edge-perspective that is normally used for the standard LDPC code ensembles.

To explain these equations, we denote n_e as the number of edge types used in the graph's ensemble and n_τ as the number of different channels over which a bit may be transmitted. $\mathbf{d} = [d_1, ..., d_{n_e}]$ is the *edge degree vector* (EDV) of length n_e, and $\mathbf{b} = [b_0, ..., b_{n_\tau}]$ is the *received degree vector* (RDV) of length $n_\tau + 1$. The vector of variables is denoted by $\mathbf{x} = [x_1, ..., x_{n_e}]$, while $\mathbf{r} = [r_o, ..., r_{n_\tau}]$ denotes the vector of variables corresponding to the received distributions. Here, $\mathbf{x}^{\mathbf{d}} = \prod_{i=1}^{n_e} x_i^{d_i}$ and $\mathbf{r}^{\mathbf{b}} = \prod_{i=0}^{n_\tau} r_i^{b_i}$. Finally, the $\upsilon_{\mathbf{b},\mathbf{d}}$ and $\mu_{\mathbf{d}}$ coefficients correspond to the fraction of variable nodes with type $(\mathbf{b}, \mathbf{d})$ and check nodes with type $(\mathbf{d})$ in the graph.

The code rate of a MET-LDPC code is given by [13]

$$R = \upsilon(\mathbf{1}, \mathbf{1}) - \mu(\mathbf{1}) \quad (12)$$

where $\mathbf{1}$ denotes a vector of all 1's where the length is determined by the context.

Based on the framework of MET-LDPC codes, the edges in the ensemble of distributed LDPC codes can be represented using three types of edge degree ($n_e = 3$) as depicted in Fig. 2. The edge degree of type 1 connects N_M variable nodes to $N_M - K_M$ check nodes (i.e. the sub-matrix of H_M). The edges in the sub-matrix H_W that connect N_W variable nodes to $N_W - K_W$ check nodes are represented using the edge degree of type 2. Finally, the edge degree of type 3 represents the edges connecting all $N_M + N_W + N_R$ variables nodes with N_R check nodes. To represent the symbols that have been received over different channels at D, we use three types of received degree ($n_\tau = 3$) to denote each received symbols, as shown in Fig. 2. With the above representation, the two multinomials of distributed LDPC codes are given by

$$\begin{aligned} \upsilon(\mathbf{r}, \mathbf{x}) \quad = \quad & r_1 \sum_{i>0}^{d_{v,1}} \sum_{k=0}^{d_{v,3}} \upsilon_{[0,1,0,0],[i,0,k]} x_1^i x_3^k + \\ & r_2 \sum_{j>0}^{d_{v,2}} \sum_{k=0}^{d_{v,3}} \upsilon_{[0,0,1,0],[0,j,k]} x_2^j x_3^k + \\ & r_3 \sum_{k>0}^{d_{v,3}} \upsilon_{[0,0,0,1],[0,0,k]} x_3^k \end{aligned} \quad (13)$$

$$\mu(\mathbf{x}) = \sum_{i>0}^{d_{c,1}} \mu_{[i,0,0]} x_1^i + \sum_{j>0}^{d_{c,2}} \mu_{[0,j,0]} x_2^j + \sum_{k>0}^{d_{c,3}} \mu_{[0,0,k]} x_3^k \quad (14)$$

where r_1, r_2 and r_3 denote the N_R, N_W and N_R received symbols from the MD, WD and RD channels, respectively. The EDV $\mathbf{d} = [i, j, k]$ represents three types of edge degree, with i, j and k denoting the variable or check nodes' degrees (sockets) of type 1, type 2 and type 3, respectively. The symbols $d_{v,l}$ and $d_{c,l}$ denote the maximum number of sockets of the type l edge degree for the variable and check nodes, respectively. We refer this code as the distributed multi-edge type LDPC (DMET-LDPC) code. The $\upsilon_{\mathbf{b},\mathbf{d}}$ coefficients that correspond to the N_M, N_W and N_R variable nodes must satisfy $\sum_{i>0}^{d_{v,1}} \upsilon_{[0,1,0,0],[i,0,k]} = T_M$, $\sum_{j>0}^{d_{v,2}} \upsilon_{[0,0,1,0],[0,j,k]} = T_W$ and $\sum_{k>0}^{d_{v,3}} \upsilon_{[0,0,0,1],[0,0,k]} = T_R$, respectively. Furthermore, the $\mu_{\mathbf{d}}$

coefficients that correspond to the $N_M - K_M$, $N_W - K_W$ and N_R check nodes must satisfy $\sum_{i>0}^{d_{c,1}} \mu_{[i,0,0]} = T_M(1 - R_{MR})$, $\sum_{j>0}^{d_{c,2}} \mu_{[0,j,0]} = T_W(1 - R_{WR})$ and $\sum_{k>0}^{d_{c,3}} \mu_{[0,0,k]} = T_R$, respectively.

Within the multinomials of the overall DMET-LDPC codes, the multinomials for the sub-matrix H_M are

$$\upsilon^M(\mathbf{r}, \mathbf{x}) = r_1 \sum_{i>0}^{d_{v,1}} \upsilon_{[0,1,0,0],[i,0,0]} x_1^i \quad (15)$$

$$\mu^M(\mathbf{x}) = \sum_{i>0}^{d_{c,1}} \mu_{[i,0,0]} x_1^i, \quad (16)$$

and the multinomials for the sub-matrix H_W are

$$\upsilon^W(\mathbf{r}, \mathbf{x}) = r_2 \sum_{j>0}^{d_{v,2}} \upsilon_{[0,0,1,0],[0,j,0]} x_2^j \quad (17)$$

$$\mu^W(\mathbf{x}) = \sum_{j>0}^{d_{c,2}} \mu_{[0,j,0]} x_2^j. \quad (18)$$

Note that the multinomials for H_M and H_W only consist of the edge degree of type 1 and type 2, respectively. Using the defined multinomials for DMET-LDPC codes, the designed rates for H, H_M and H_W are given by

$$R_T = \frac{K_M + K_W}{N} = 1 - \sum_{i>0}^{d_{c,1}} \mu_{[i,0,0]} - \sum_{j>0}^{d_{c,2}} \mu_{[0,j,0]} - \sum_{k>0}^{d_{c,3}} \mu_{[0,0,k]} \quad (19)$$

$$R_{MR} = \frac{K_M}{N_M} = \frac{\sum_{i>0}^{d_{v,1}} \upsilon_{[0,1,0,0],[i,0,0]} - \sum_{i>0}^{d_{c,1}} \mu_{[i,0,0]}}{\sum_{i>0}^{d_{v,1}} \upsilon_{[0,1,0,0],[i,0,0]}} \quad (20)$$

$$R_{WR} = \frac{K_W}{N_W} = \frac{\sum_{j>0}^{d_{v,2}} \upsilon_{[0,0,1,0],[0,j,0]} - \sum_{j>0}^{d_{c,2}} \mu_{[0,j,0]}}{\sum_{j>0}^{d_{v,2}} \upsilon_{[0,0,1,0],[0,j,0]}}. \quad (21)$$

V. Design of DMET-LDPC codes

The DMET-LDPC codes must be constructed so that successful decoding is guaranteed at the destination and relay. Hence, the code design involves the constructions of three codes: the two subcodes corresponding to the two sub-parity check matrices H_M and H_W, and the overall distributed LDPC codes given by the parity check matrix H in (11). We execute the code design in two steps. In the first step, we optimize the two sub-parity check matrices separately so that H_M and H_W are optimum in the sense that both subcodes are achieving the capacities for the MR and WR channels, respectively. In the second step, we design the parity check matrix H by fixing the optimized H_M and H_W obtained in the first step and optimize the other three sub-matrices H_{1R}, H_{2R} and H_{3R}.

The design of H_M or H_W can be performed conventionally using a standard optimization method for the irregular LDPC codes in a point-to-point channel, since both matrices are only represented via one edge type, i.e. edge degree of type 1 for H_M and edge degree of type 2 for H_W. To ensure that the relay can successfully decode the two subcodes, H_M and H_W must be designed at $R_{MR} \leq C(\gamma_{MR})$ and $R_{WR} \leq C(\gamma_{MR})$, respectively. The optimized node perspective degree distribution pair of a standard irregular LDPC code consists of the polynomials $\mathring{\lambda}(x) = \sum_{i>0}^{d_{v,1}} \mathring{\lambda}_i x^{i-1}$ and $\mathring{\rho}(x) = \sum_{i>0}^{d_{v,1}} \mathring{\rho}_i x^{i-1}$, where the coefficient of $\mathring{\lambda}_i$ and $\mathring{\rho}_i$ are non-negative real numbers satisfying $\sum_{i>0}^{d_{v,1}} \mathring{\lambda}_i = 1$ and $\sum_{i>0}^{d_{v,1}} \mathring{\rho}_i = 1$, respectively. The relationships between the optimized conventional node-perspective degree distributions of H_M and H_W and the multi-edge type multinomials are given as follow

$$\upsilon^t(\mathbf{r}, \mathbf{x}) = r_s \cdot \mathring{\lambda}^t(x) \cdot T_i \quad (22)$$

$$\mu^t(\mathbf{x}) = \mathring{\rho}^t(x) \cdot T_i(1 - R_{iR}) \quad (23)$$

where $(\mathring{\lambda}^t(x), \mathring{\rho}^t(x))$ is the optimized degree distribution pairs for H_t, with $t \in \{M, W\}$. r_s corresponds to the received variable. If $t = M$ $(t = W)$, the received variable is r_1 (r_2).

After we design H_M and H_W and then convert them into the MET-LDPC multinomials, the next step is to design the overall distributed codes (i.e. H) at rate R_T. The design of H is performed by optimizing the ensembles of H_{1R}, H_{2R} and H_{3R}, while fixing the optimized ensembles of H_M and H_W. Here, the searching for the best H is only performed within the edge degree of type 3. Since different segment of the received symbols at D have been transmitted over different channels and experience different SNRs, the optimization of H at rate R_T involves maximizing the noise thresholds for the MD, WD and RD channels simultaneously. This optimization problem is equal to minimizing the average received SNR needed to successfully decode the overall distributed codeword at D. We define σ_{N_M}, σ_{N_W} and σ_{N_R} as the noise standard deviations for the N_M, N_W and N_R variables, respectively. That is, the corresponding received SNRs for MD, WD and RD channels are $\rho_{MD} = 1/\sigma_{N_M}^2$, $\rho_{WD} = 1/\sigma_{N_M}^2$ and $\rho_{RD} = 1/\sigma_{N_R}^2$, respectively. With these definitions, the average noise standard deviation (threshold) for the overall $N_M + N_W + N_R$ distributed codeword is given by

$$\sigma_{sys} = \sqrt{\frac{1}{\frac{T_M}{\sigma_{N_M}^2} + \frac{T_W}{\sigma_{N_W}^2} + \frac{T_R}{\sigma_{N_R}^2}}} \quad (24)$$

where T_M, T_W and T_R are the percentage of N_M, N_W and N_R symbols in the overall Tanner graph.

We optimize the ensemble of H at rate R_T using differential evolution [14], by maximizing the average σ_{sys} under the multi-edge type density evolution [13], subject to $\sigma_{N_M} \leq \sigma_{MD_{lim}}$, $\sigma_{N_W} \leq \sigma_{WD_{lim}}$ and $\sigma_{N_R} \leq \sigma_{RD_{lim}}$, where $\sigma_{MD_{lim}}$, $\sigma_{WD_{lim}}$ and $\sigma_{RD_{lim}}$ are the theoretical limits for the MD, WD and RD Gaussian channels, respectively. Below we outline the procedure to generate the set of type 3 edge multinomials of H_{1R}, H_{2R} and H_{3R}. This procedure initializes the optimization process of the differential evolution.

1) For each optimized coefficient $\upsilon_{[0,1,0,0],[L_1,0,0]}$ of the N_M variable nodes, with L_1 type 1 sockets, we generate an integer of k_1 type 3 socket(s) to obtain $\upsilon_{[0,1,0,0],[L_1,0,k_1]}$. The same procedure is applied for each optimized coefficient $\upsilon_{[0,0,1,0],[0,L_2,0]}$ of the N_W

variable nodes, with L_2 type 2 sockets, where we generate an integer of k_2 type 3 socket(s) to obtain $\upsilon_{[0,0,1,0],[0,L_2,k_2]}$. Then, we also generate an integer of k_3 type 3 socket(s) to obtain $\upsilon_{[0,0,0,1],[0,0,k_3]}$ for the N_R variable nodes. At this stage, we have successfully generated the variable node's multinomial in (13).

2) We compute the total number of type 3 edges (sockets) for the generated variable node's multinomial using

$$\upsilon_{x_3}(\mathbf{1},\mathbf{1}) = \frac{d}{d_{x_3}}\upsilon(\mathbf{r},\mathbf{x}). \quad (25)$$

3) The average value of the check node degrees for $[H_{1R}\ H_{2R}\ H_{3R}]$ can be calculated using

$$d_{c,ave} = \frac{\upsilon_{x_3}(\mathbf{1},\mathbf{1})}{T_R}. \quad (26)$$

4) We generate a concentrated type 3 edge multinomial for the check nodes. This multinomial is given by

$$\sum_{k>0}^{d_{c,3}} \mu_{[0,0,k]} x_3^k = \mu_{[0,0,d_{c,3}]} \cdot x_3^{d_{c,3}} + \mu_{[0,0,d_{c,3}-1]} \cdot x_3^{d_{c,3}-1} \quad (27)$$

where $d_{c,3} = \lceil d_{c,ave} \rceil$ and $\lceil \cdot \rceil$ represents the ceiling function. The values of $\mu_{[0,0,d_{c,3}]}$ and $\mu_{[0,0,d_{c,3}-1]}$ are obtained by solving the following two equations

$$T_R = \mu_{[0,0,d_{c,3}]} + \mu_{[0,0,d_{c,3}-1]} \quad (28)$$

$$\upsilon_{x_3}(\mathbf{1},\mathbf{1}) = d_{c,3} \cdot \mu_{[0,0,d_{c,3}]} + (d_{c,3}-1) \cdot \mu_{[0,0,d_{c,3}-1]}. \quad (29)$$

5) At the end of Step 4, we have successfully generated the check node's multinomials in (14). The noise threshold of the generated variable and check nodes' multinomials of H can then be computed using the multi-edge type density evolution [13].

After all the multinomials in the initial set (population) are generated, the process of mutation, recombination and selection through the differential evolution algorithm is performed to find the optimum H.

VI. Numerical Results

We evaluate the performance of our proposed methodology by designing the distributed LDPC codes for the following asymmetric TD-MARC.

- The BAWGN capacities for each link are given as follows, $C(\rho_{MR}) = 0.8$, $C(\rho_{WR}) = 0.9$, $C(\rho_{MD}) = 0.4$, $C(\rho_{WD}) = 0.45$ and $C(\rho_{RD}) = 0.85$.
- The rate ratio between the two users is $\varphi = 0.5$.
- Using (7) and (8), the optimal values of time sharing parameters that maximize the overall rate for the distributed LDPC code are $T_M = 0.465$, $T_W = 0.2067$ and $T_R = 0.3283$.
- The designed rate for the two subcodes corresponding to H_M and H_W are $R_{MR} = 0.8$ and $R_{WR} = 0.9$. The optimum overall rate is $R_T = 0.55803$.

The optimization problem of the distributed LDPC codes involves designing three codes of the parity check matrices H_M, H_W and H at the designed rates of R_{MR}, R_{WR} and R_T, respectively. In this work, the optimized degree distributions for H_M and H_W are obtained from [15], which are given by the following node-perspective degree distribution pairs: $(\mathring{\lambda}^M(x) = 0.39391x + 0.36389x^2 + 0.12012x^5 + 0.01303x^6 + 0.10905x^{12}$, $\mathring{\rho}^M(x) = x^{19})$, and $(\mathring{\lambda}^W(x) = 0.35855x + 0.41782x^2 + 0.18321x^7 + 0.04042x^8$, $\mathring{\rho}^W(x) = x^{37})$. The asymptotic performances for the optimized H_M and H_W are 0.13882 dB and 0.14543 dB, respectively.

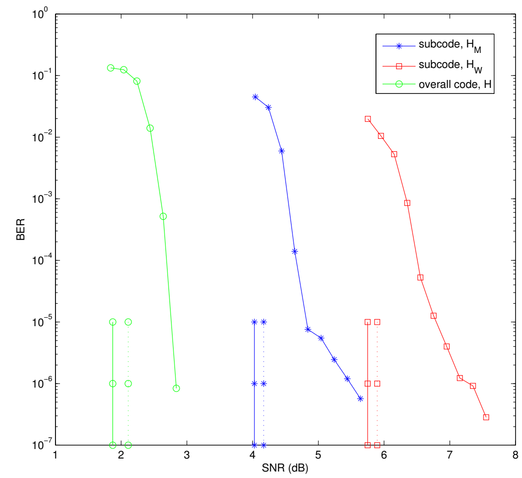

Fig. 3. BER performances for H_M, H_W and H designed at $R_M = 0.8$, $R_W = 0.9$ and $R_T = 0.55803$, respectively, for the optimum time sharing parameters of $T_M = 0.465$, $T_W = 0.2067$ and $T_R = 0.3283$. The dotted lines represent the optimized thresholds for the codes and the theoretical limits for the correspondence designed rates.

Now, we discuss the design of the overall distributed LDPC code defined by the parity check matrix H at $R_T = 0.55803$. To begin the optimization of the overall distributed LDPC code, we apply (22) and (23) to convert the optimized degree distributions of H_M and H_W into the multi-edge type multinomials with edge degree of types 1 and 2, respectively. Fixing these optimized multinomials, we design the overall distributed LDPC code by optimizing the ensemble of type 3 edge degree between $N_M + N_W + N_R$ variable nodes and N_R additional parity check nodes using the methodology of section V. The optimized distributed LDPC code denoted by Code 1 is given in Table I. It performs asymptotically within 0.2353 dB from the theoretical limit, i.e. the code achieves the thresholds of $\sigma^*_{N_M} = 1.12$ for the N_M variables transmitted through the MD link, $\sigma^*_{N_W} = 1.058$ for the N_W variables transmitted through the WD link and $\sigma^*_{N_R} = 0.554$ for the N_R variables transmitted through the RD link. In Table I, the fraction of the RDV of type 1, type 2 and type 3 are $\sum_{r=1} \upsilon_{[0,r,0,0],[i,j,k]} = 0.465$, $\sum_{r=1} \upsilon_{[0,0,r,0],[i,j,k]} = 0.2067$, and $\sum_{r=1} \upsilon_{[0,0,0,r],[i,j,k]} = 0.3283$, respectively, which correspond to the fraction of the N_M, N_W and N_R variables in the overall Tanner graph satisfying the optimized time sharing

CODE 1												
RDV				EDV								
Type 0	Type 1	Type 2	Type 3	$\upsilon_{[i,j]}$	Type 1	Type 2	Type 3	$\mu_{[i,j]}$	Type 1	Type 2	Type 3
0	1	0	0	0.051934	2	0	0	0.093	20	0	0
0	1	0	0	0.062112	2	0	3	0.02067	0	38	0
0	1	0	0	0.069124	2	0	1	0.140847	0	0	9
0	1	0	0	0.073365	3	0	0	0.187453	0	0	10
0	1	0	0	0.063673	3	0	6				
0	1	0	0	0.032169	3	0	2				
0	1	0	0	0.038670	6	0	5				
0	1	0	0	0.017187	6	0	25				
0	1	0	0	0.006060	7	0	1				
0	1	0	0	0.025110	12	0	1				
0	1	0	0	0.025596	12	0	5				
0	0	1	0	0.074113	0	2	1				
0	0	1	0	0.004605	0	3	0				
0	0	1	0	0.081759	0	3	2				
0	0	1	0	0.013631	0	8	24				
0	0	1	0	0.024237	0	8	7				
0	0	1	0	0.008355	0	9	6				
0	0	0	1	0.245988	0	0	2				
0	0	0	1	0.014986	0	0	3				
0	0	0	1	0.067326	0	0	5				

R_{MR} , R_{WR}, R_T	0.8 , 0.9, 0.55803
σ^*_{MR} , σ^*_{WR}, σ^*_{sys} ($\sigma^*_{N_M}$, $\sigma^*_{N_W}$, $\sigma^*_{N_R}$)	0.615531 , 0.50723 , 0.78446 (1.12, 1.058, 0.554)
gap_{MR} , gap_{WR}, gap_{sys}	0.13382 dB , 0.14543 dB, 0.2353 dB

TABLE I

THE OPTIMIZED MULTINOMIALS FOR DMET-LDPC CODE DESIGNED AT $R_{MR} = 0.8$, $R_{WR} = 0.9$ AND $R_T = 0.55803$ WITH OPTIMAL TIME ALLOCATION OF $T_M = 0.465$, $T_W = 0.2067$ AND $T_R = 0.3283$.

parameters.

To validate the asymptotic performances of the optimized codes, we simulate the finite-length performances of Code 1 with a total block length of $N = 20000$ bits. The block length of the two subcodes are $N_M = 9300$ bits and $N_W = 4134$ bits, while the block length for the additional parity check bits from the relay is $N_R = 6566$ bits. The BER curves for Code 1 is given in Fig. 3. At an error probability of 10^{-5}, the two subcodes H_M and H_W perform around 0.75 dB and 1 dB from the theoretical limits, respectively. The overall distributed code H performs around 0.875 dB from the theoretical limits.

VII. Conclusion

We have extended the general concept of irregular LDPC codes, captured by MET-LDPC codes, to the construction of distributed LDPC codes for TD-MARC. A new design methodology has been proposed, where we transform the design of distributed LPDC codes into a problem of optimizing MET-LDPC codes. In order to optimize the codes under density evolution, we formulate the average threshold for the proposed code design. This is because the received symbols within a single distributed codeword have been transmitted through different channels and experience different SNRs.

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