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Optimization for Pragmatic Half-Duplex Relay Network

Marwan H. Azmi, Jun Li, Robert Malaney and Jinhong Yuan

School of Electrical Engineering and Telecommunications
University of New South Wales, Sydney, NSW 2052, AUSTRALIA
Email: marwan@student.unsw.edu.au, jun.li@unsw.edu.au, r.malaney@unsw.edu.au, j.yuan@unsw.edu.au

***Abstract*—In relay networks, we may not possess the ability to tune all system parameters in order to achieve the maximum achievable rate promised by theoretical analysis. This paper investigates the pragmatic issue of sub-optimal relay networks, where we can only optimize either on the time allocation, or on the power allocation. Our study concludes that optimizing on the power allocation, or on the time allocation, can achieve more than 95% of the relaying gain relative to the optimization of both time and power simultaneously. To produce the result, we derive the closed-form expression of the optimum power allocation for the source and relay that obtains the achievable rate of the half-duplex relay channel with fixed (equal) time allocation. We also derive the closed-form expression of the optimum time allocation between the source and relay transmission that obtains the achievable rate of the half-duplex relay channel with fixed power allocation. We demonstrate that for small SNR, where relaying is most advantageous, almost all relaying gain can be achieved by only optimizing the power allocation. Conversely, optimizing the time allocation alone is sufficient to achieve most of the relaying gain when the system's SNR is large. This result is important for pragmatic designs of emerging relay communication systems.**

I. Introduction

Previous studies have shown that cooperative diversity [1] in the relay networks provides spatial diversity and can potentially increase the transmission rate of communication systems. The relay channel was first introduced by Van der Meulen [2]. The fundamental limit of the relay channel has been widely studied for full-duplex [3], [4] and half-duplex [5], [6], [8] relaying. Although full-duplex relaying can achieve higher transmission rates, full-duplex nodes are difficult to be realized in practice due to the constraint of shielding and accurate interference cancelation between the transmitted and the received signals. The simpler solution is for the relay system to operate in the half-duplex mode in which the nodes in the network do not transmit and receive simultaneously.

In [5], the general expression for the achievable rate of the half-duplex Gaussian relay channel is derived. From the information-theoretic point of view, this achievable rate is a function of various system parameters. These parameters are the correlation between the signal transmitted by the source and the relay, the fraction of time in which the relay receives and transmits, and the power allocation between the source and the relay. The effect of these parameters except for the effect of the power allocation to the system's performance has been studied in [6], where the parameters that maximize the achievable rate are found using a numerical search. In [7], the problem of finding the optimum power allocation that maximizes the throughput has been studied for both full-duplex and half-duplex relay channel in Rayleigh fading environment and the procedure to numerically compute the optimum power allocation is proposed.

In both Gaussian [6] and Rayleigh fading [7] environments, the maximum achievable rate of half-duplex relay channel is achieved with a specific combination of all system parameters. In practice, relay system may not possess the ability of tuning all system parameters in order to achieve the highest throughput offers by the theoretical limit, e.g. it is a challenge to design arbitrarily correlated source and relay codebook.

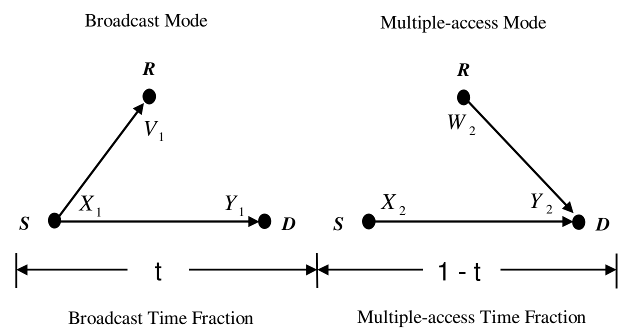

Figure 1. Half-duplex relay channels.

Our study is motivated by the pragmatic issue of sub-optimal half-duplex Gaussian relay networks, where we can only optimize either on the time allocation, or on the power allocation. Instead of finding the optimum parameter (time or power) numerically, we derive the closed-form expression of the optimum power allocation for the source and relay that obtains the achievable rate of the half-duplex relay channel with fixed (equal) time allocation. We also derive the closed-form expression of the optimum time allocation between the source and relay transmission that obtains the achievable rate of the half-duplex relay channel with fixed power allocation. We then compare the two achievable rates of the sub-optimal relay networks with the optimal relay network when both time and power are jointly optimized. Depending on the signal-to-noise ratio (SNR), we conclude that optimizing on the power

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allocation, or on the time allocation, is sufficient to achieve more than 95% of the relaying gain relative to the optimization of both time and power simultaneously. Our result is important from a pragmatic point of view, as it can lead to simpler designs for relay communication systems.

II. Channel model

We consider the half-duplex (time division) relay channel shown in Fig. 1, where one source (S) sends independent messages to the destination (D), and in doing so it is aided by one half-duplex relay (R). Given a time window L , the source broadcasts its information for a fraction of the time tL (first time slot), which can be received by both R and D. Due to the nature of broadcast channel, we call the first time slot the broadcast (BC) mode. In the remaining fraction of the time $t'L = (1-t)L$ (second time slot), the source and the relay transmit their signal to the destination simultaneously, which form the multiple-access channel between the source, relay and destination. We refer the second time slot as the multiple-access (MAC) mode.

In this paper, we denote the source transmitted signal, relay received signal, relay transmitted signal and destination received signal by using X, V, W and Y (see Fig. 1), respectively. In order to distinguish between the transmission modes, we denote the BC mode using subscript 1 and the MAC mode using subscript 2. With the above notations, the half-duplex relay channel is defined by

$$V_1 = h_{SR}X_1 + N_{R1} \quad (1)$$
$$Y_1 = h_{SD}X_1 + N_{D1} \quad (2)$$
$$Y_2 = h_{SD}X_2 + h_{RD}W_2 + N_{D2} \quad (3)$$

where, h_{ij} is the channel realization between node i and node j, while N_{D1} and N_{D2} are the noise realizations at the destination in BC and MAC modes, respectively, and N_{R1} is the noise realization at the relay in BC mode. All the noises are Gaussian with zero mean and unit variance. We consider static-one dimensional additive white Gaussian noise (AWGN); however, extension to circularly symmetric AWGN channels is straightforward. Perfect global channel knowledge is assumed at all nodes.

An average global transmission power is imposed on the nodes, denoted by the symbol Θ,

$$\Theta : tP_{S_{BC}} + (1-t)(P_{S_{MAC}} + P_{R_{MAC}}) \leq P \quad (4)$$

where $P_{S_{MAC}}$ and $P_{R_{MAC}}$ are the source and relay transmission power in the MAC mode, respectively, while $P_{S_{BC}}$ is the source transmission power in BC mode. P is the total transmission power in the system and it is equivalent to the SNR in our plots since the noise power is normalized to unity.

The relay position is described in Fig. 2, where S, R and D lie on a straight line. We normalized the distance between S and D to unity and d denotes the position of R relative to S. With this setup, the SD, SR and RD channel gains are $\gamma_{sd} = 1$, $\gamma_{sr} = \frac{1}{d^\alpha}$ and $\gamma_{rd} = \frac{1}{(1-d)^\alpha}$, respectively, where α is the path-loss exponent. The result in this work is independent of α. In this paper, we choose $\alpha = 2$ for our plots.

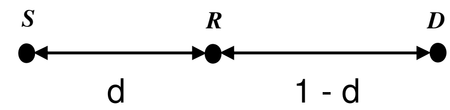

Figure 2. The geometry of the nodes in relay channels.

Remark 1: Even though we assume all the nodes lie on a straight line, Theorem 2 presented in this paper is valid for any topology of relay channels. Theorem 1 is valid for any topology of relay channels where the distance between S-D is less than R-D, i.e. the relay is closer to the destination.

III. Achievable rate of Decode-and Forward for half-duplex relay channel

We review the known achievable rate of decode-and-forward (DF) protocol for the general half-duplex relay channels, which is given by [5], [6]

$$R_{DF} = \sup_{0\leq t\leq 1} \min \{tI(X_1;V_1) + t'I(X_2;Y_2|W_2), \\ tI(X_1;Y_1) + t'I(X_2,W_2;Y_2)\}. \quad (5)$$

For the AWGN channel, the achievable rate of DF can be written as [6]

$$R_{DF_{AWGN}} = \sup_{\Theta,0\leq t,\rho\leq 1} \min \{tC(P_{SR}) \\ + t'C((1-\rho^2)P_{SD2}) \text{ , } tC(P_{SD1}) \\ + t'C(P_{SD2} + P_{RD} + 2\rho\sqrt{P_{SD2}P_{RD}})\} \quad (6)$$

where $C(x) = \frac{1}{2}\log_2(1+x)$ and ρ is the correlation between the source's and the relay's signals in MAC mode. The notations used in (6) are defined as

$$P_{SR} = P_{S_{BC}}\gamma_{sr} \quad ; \quad P_{SD1} = P_{S_{BC}}\gamma_{sd} \\ P_{RD} = P_{R_{MAC}}\gamma_{rd} \quad ; \quad P_{SD2} = P_{S_{MAC}}\gamma_{sd}. \quad (7)$$

In this work, we only consider a full correlation ($\rho = 1$) signals between source and relay in MAC mode, i.e. the source and relay transmit the same signal (information). The achievable rate of DF when $\rho = 1$ is given by

$$R_{DF1} = \sup_{\Theta,0\leq t\leq 1} \min \{tC(P_{SR}) \text{ , } tC(P_{SD1}) + \\ t'C(P_{SD2} + P_{RD} + 2\sqrt{P_{SD2}P_{RD}})\}. \quad (8)$$

Now, the two parameters that determine the achievable rate of DF in (8) are the time fraction t and the power constraint Θ.

A. Case 1: Optimum time and equal power allocations

The optimum time fraction t with fixed power allocation between the nodes in the half-duplex relay channels can be chosen such that R_{DF1} is maximized:

$$t^* = arg \max_{0\leq t\leq 1} R_{DF1}. \quad (9)$$

Theorem 1: For equal power allocation, $P_{S_{BC}} = P$, $P_{S_{MAC}} = P/2$ and $P_{R_{MAC}} = P/2$, the DF achievable rate of the half-duplex relay channel with the optimum time fraction, t^*, is

$$R_{DF1}(t^*) = \frac{C_{SR}\, C_{MAC}}{C_{SR} + C_{MAC} - C_{SD}} \quad (10)$$

where

$$C_{SR} = C(P_{SR}) \quad (11)$$
$$C_{SD} = C(P_{SD1}) \quad (12)$$
$$C_{MAC} = C(P_{SD2} + P_{RD} + 2\sqrt{P_{SD2}P_{RD}}). \quad (13)$$

Proof: The first minimization term tC_{SR} in (8) is monotonically increasing with t and the second minimization term $tC_{SD} + (1-t)C_{MAC} = t(C_{SD} - C_{MAC}) + C_{MAC}$ is monotonically decreasing with t since $C_{MAC} > C_{SD}$. So the maximization of (9) can be found at the cross-over point of the two terms and the optimal time sharing parameter can be obtained by solving the following equation:

$$t^* C_{SR} = t^* C_{SD} + (1 - t^*)C_{MAC} \quad (14)$$

which leads to the optimal value of time sharing

$$t^* = \frac{C_{MAC}}{C_{SR} + C_{MAC} - C_{SD}}. \quad (15)$$

The DF achievable rate with the optimum time sharing parameter is $t^* C_{SR}$, which gives equation (10) and completes the proof [1]. ■

B. Case 2: Optimum power and equal time allocations

Now, we consider the case where the time allocation is fixed with equal time sharing, $t = 1/2$. The achievable rate for DF can be simplified as

$$R_{DF1} = \frac{1}{2} \sup_{\Theta} \min\{C_{SR}\ ,\ C_{SD} + C_{MAC}\}. \quad (16)$$

The average power constraint in (4) can now be written as

$$\Theta : P_{BC} + P_{MAC} \le 2P \quad (17)$$

with $P_{BC} = P_{S_{BC}}$ denotes the total transmission power in BC mode and $P_{MAC} = P_{S_{MAC}} + P_{R_{MAC}}$ is the total transmission power in MAC mode.

To solve (16), we first find the optimum power allocation in MAC mode, i.e., the optimum power allocation for $P_{S_{MAC}}$ and $P_{R_{MAC}}$ that maximizes C_{MAC}. Mathematically, this maximization problem is defined by

$$z = \max \quad (\sqrt{P_{R_{MAC}}\gamma_{rd}} + \sqrt{P_{S_{MAC}}\gamma_{sd}}) \quad (18)$$

subject to constraint $P_{MAC} = P_{S_{MAC}} + P_{R_{MAC}}$. By utilizing Lagrange Multipliers, the optimum $P^*_{S_{MAC}}$ and $P^*_{R_{MAC}}$ can be found as

$$P^*_{S_{MAC}} = \frac{\gamma_{sd}}{\gamma_{sd} + \gamma_{rd}} P_{MAC} \quad (19)$$
$$P^*_{R_{MAC}} = \frac{\gamma_{rd}}{\gamma_{sd} + \gamma_{rd}} P_{MAC} \quad (20)$$

[1]The proof is similar to the case of half-duplex relay channel when only relay sends information and source is silent in the second time slot [9].

By replacing the optimum power allocation in MAC mode into (7), the optimum C^*_{MAC} in (13) is given by

$$C^*_{MAC} = C(P_{MAC}(\gamma_{sd} + \gamma_{rd})). \quad (21)$$

Now, the problem of finding the optimal power allocation that maximizes (16) can be written as

$$R_{DF1} = \frac{1}{4} \sup_{0 \le P_{BC} \le 2P} \min\{\ \log_2(1 + \gamma_{sr}P_{BC})\ ,\ \log_2(1 + \gamma_{sd}P_{BC}) + \log_2(1 + (2P - P_{BC})(\gamma_{sd} + \gamma_{rd}))\ \}. \quad (22)$$

Note that, we only need to find the optimum value of P^*_{BC} that maximizes (22). To find P^*_{BC}, we define the following two functions

$$f_1(P_{BC}) = 1 + \gamma_{sr}P_{BC} \quad (23)$$
$$f_2(P_{BC}) = (1+\gamma_{sd}P_{BC})(1+(2P-P_{BC})(\gamma_{sd}+\gamma_{rd})) \quad (24)$$

The maximization problem in (22) is similar to solving the following maximization problem

$$y = \sup_{0 \le P_{BC} \le 2P} \min\{f_1\ ,\ f_2\}. \quad (25)$$

Function f_1 is a monotonically increasing with P_{BC} since $\gamma_{sr} > 0$, while function f_2 is a concave function with P_{BC} and it can be expanded into

$$f_2(P_{BC}) = -A_2 P_{BC}^2 + B_2 P_{BC} + C_2 \quad (26)$$

with

$$A_2 = \gamma_{sd}(\gamma_{sd} + \gamma_{rd}) \quad (27)$$
$$B_2 = 2P\gamma_{sd}(\gamma_{sd} + \gamma_{rd}) - \gamma_{rd} \quad (28)$$
$$C_2 = 2P\gamma_{sd}(\gamma_{sd} + \gamma_{rd}) + 1. \quad (29)$$

Since $A_2 > 0$, it is straightforward to prove that $f_2(P_{BC})$ is indeed a concave function because $\frac{d^2 f_2}{dP_{BC}^2} < 0$. The optimum value of P_{BC} that maximizes f_2 is computed by setting $\frac{df_2}{dP_{BC}} = 0$, which gives

$$P_{BC_1} = P - \frac{\gamma_{rd}}{2\gamma_{sd}(\gamma_{sd} + \gamma_{rd})}. \quad (30)$$

The cross-over point between f_1 and f_2 can be obtained by solving $f_1 = f_2$. Since f_2 is a concave function, there is a maximum of two cross-over points, which can be computed using

$$P_{BC_2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (31)$$

where

$$A = -\gamma_{sd}(\gamma_{sd} + \gamma_{rd}) \quad (32)$$
$$B = 2P\gamma_{sd}(\gamma_{sd} + \gamma_{rd}) - (\gamma_{sr} + \gamma_{rd}) \quad (33)$$
$$C = 2P\gamma_{sd}(\gamma_{sd} + \gamma_{rd}). \quad (34)$$

Since $A < 0$ and $B < \sqrt{B^2 - 4AC}$, the cross-over point within the power allocation range of $0 \le P_{BC} \le 2P$ is given by

$$P_{BC_2*} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \quad (35)$$

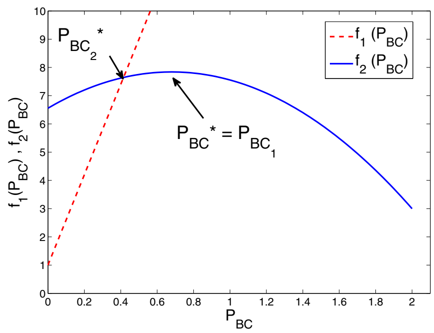

Figure 3. Functions $f_1(P_{BC})$ and $f_2(P_{BC})$ for $P = 1$ and $d = 0.25$. The optimum P^*_{BC} is P_{BC_1}.

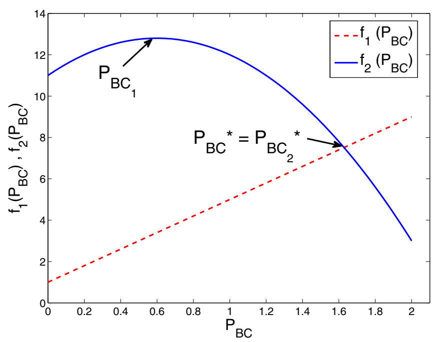

Figure 4. Functions $f_1(P_{BC})$ and $f_2(P_{BC})$ for $P = 1$ and $d = 0.5$. The optimum P^*_{BC} is $P_{BC_2}*$.

Theorem 2: For $P_{BC} \in [0, 2P]$ and $t = 1/2$, the optimum P^*_{BC} that maximizes the achievable rate of DF for the half-duplex relay channel in (22) is

$$P^*_{BC} = \begin{cases} P_{BC_1}, & \text{for } P_{BC_1} > P_{BC_2}* \\ P_{BC_2}*, & \text{for } P_{BC_2}* > P_{BC_1} \end{cases} \quad (36)$$

Proof: Within $P_{BC} \in [0, 2P]$, there is only one cross-over point between f_1 and f_2 given by $P_{BC_2}*$ in (35). To prove this statement, we evaluate function f_1 and f_2 at $P_{BC} = 0$ and $P_{BC} = 2P$. We can show that

$$f_2 > f_1, \quad \text{for } P_{BC} = 0 \quad (37)$$
$$f_2 < f_1, \quad \text{for } P_{BC} = 2P \quad (38)$$

Given (37) and (38), both functions f_1 and f_2 can only intersect with each other once at $P_{BC_2}*$ for $P_{BC} \in [0, 2P]$, since f_1 is monotonically increasing with P_{BC} and f_2 is a concave function.

The optimum P^*_{BC} can be found by evaluating the two regions: $(0 \leq P_{BC} \leq P_{BC_2}*)$ and $(P_{BC_2}* \leq P_{BC} \leq 2P)$. In the first region, y is lower bounded by f_1. Since f_1 is monotonically increasing with P_{BC}, the optimum P^*_{BC} in this region will always be $P_{BC_2}*$ (at the boundary). In the second region, there are two scenarios of optimum P^*_{BC}, as y is now lower bounded by f_2. If $(P_{BC_2}* \leq P_{BC_1} \leq 2P)$, i.e., P_{BC_1} exists in the second region, the optimum P^*_{BC} is P_{BC_1} since f_2 is a concave function. For the second scenario, the optimum P^*_{BC} is $P_{BC_2}*$ if $(P_{BC_1} \leq P_{BC_2}*)$, which is similar to the first region. This completes the prove. ■

The examples of the optimum P^*_{BC} for the two cases given in Theorem 2 are illustrated in Fig. 3 and 4.

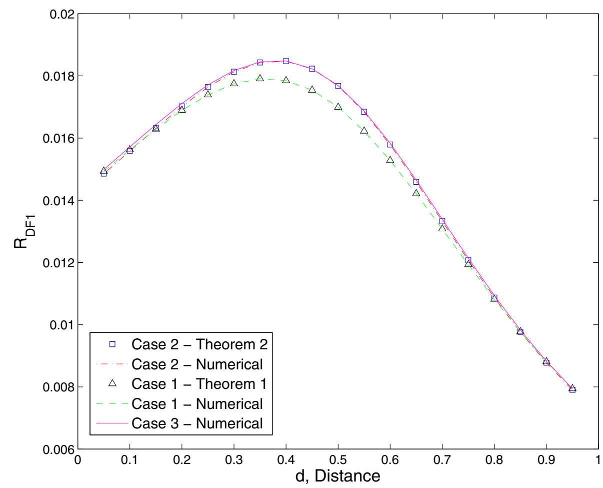

Figure 5. The DF achievable rates of half-duplex relay channel for $P = 0.01$.

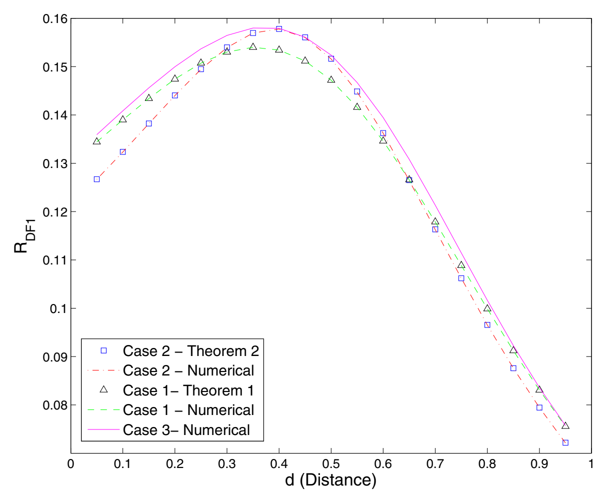

Figure 6. The DF achievable rates of half-duplex relay channel for $P = 0.1$.

IV. Results

The achievable rates of DF (Cases 1 and 2) for the half-duplex Gaussian relay channel are given in Fig. 5-8 for different values of P (SNR) and d (the distance between S and R). To verify the result in Theorem 1 and Theorem 2, we numerically search within the entire possible range of the time fraction for Case 1 (with a fixed power allocation) and within

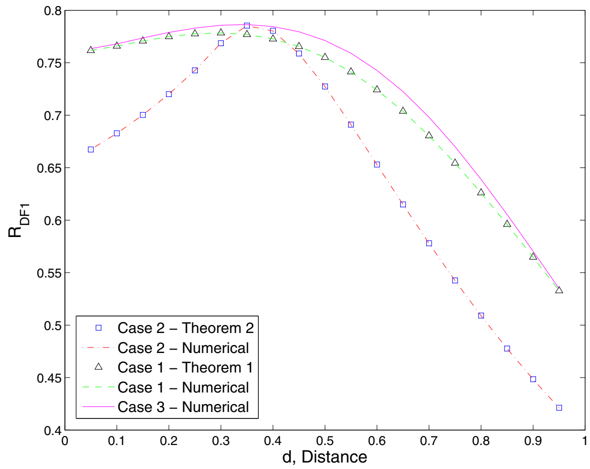

Figure 7. The DF achievable rates of half-duplex relay channel for $P = 1$.

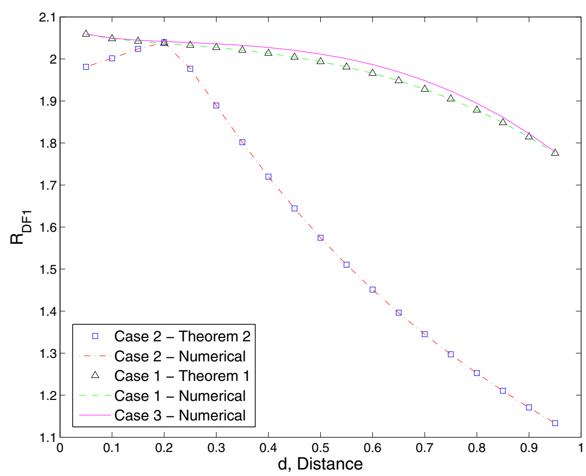

Figure 8. The DF achievable rates of half-duplex relay channel for $P = 10$.

the entire possible range of the power constraint for Case 2 (with equal time slot) to obtain the optimum rates for the two cases. Then, we compare the optimum numerical achievable rates with the achievable rates computed using Theorem 1 and Theorem 2. In all the figures, the two achievable rates (numerical and analytical) match with the each other. We also perform an extensive numerical search within all possible range of $t \in [0, 1]$ and $P_{BC} \in [0, 2P]$ to jointly optimize the time and power allocations in order to maximize the DF rate, which we refer to as Case 3. We pose the following question: How much do we lose if we can only optimize the time allocation as in Case 1 or the power allocation as in Case 2 instead of having the optimum values for both parameters as in case 3? From the figures, we can observe that:

1) As SNR decreases (see Figures 6 and 5), Case 2 achieves a higher DF rate than Case 1, and Case 2 performs very close to Case 3 at a very low SNR (see Fig. 5).
2) As SNR increases (see Figures 7 and 8), Case 1 achieves a higher DF rate than Case 2, and Case 1 performs very close to Case 3 at a very large SNR (see Fig. 8).

From these observations, we can say that it is sufficient to only optimize one parameter (time or power) in order to maximize the achievable rate of half-duplex relay channel. Depending on the SNR, the performance lost by only optimizing either on the time allocation, or on the power allocation, is very small comparing to the case where we jointly optimize both parameters (time and power). For example, we consider the achievable rate of the half-duplex relay channel in Fig. 6. For a particular relay position d, the difference between the achievable rate of any two cases can be computed. Taking the average of this achievable rate different for all d positions, we can compare the performance of the two cases. On average, Case 1 and Case 2 can gain around 97.7% and 96.9%, respectively, of the achievable rate of Case 3.

V. Conclusions

We have derived the closed-form expression of the optimum power allocation with equal time allocation ($t = 1/2$) and the optimum time allocation with a fixed power allocation for the correlated half-duplex relay channel ($\rho = 1$). Our observations conclude that optimizing on the power allocation, or on the time allocation, can achieve more than 95% of the relaying gain relative to the optimization of both time and power simultaneously. For small SNRs, where relaying is most advantageous, most relaying gain can be achieved by only optimizing the power allocation, while optimizing the time allocation alone is sufficient to achieve most of the relaying gain when the system's SNR is large. This result can lead to simpler designs of emerging relay communication systems.

Acknowledgement

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